SHAPE AND MATERIAL PARAMETER RECONSTRUCTION OF AN ISOTROPIC or ANISOTROPIC SOLID IMMERSED IN A FLUID

Inverse Problems for PDEs, Magique 3D, June of 2016

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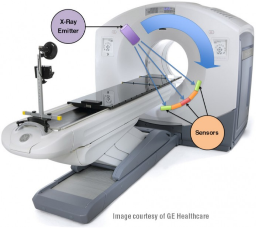
** CSUN & INRIA, Associate Team Magic, USA

Motivation and context

APPLICATIONS

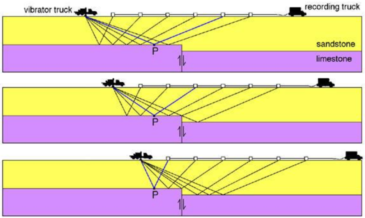
(a) Obstetric ultrasonography



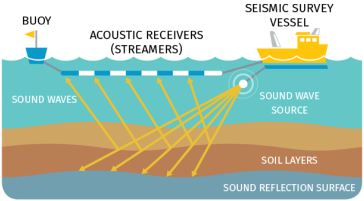
(b) X-ray computed tomography

Motivation and context

APPLICATIONS



(a) Terrestrial seismic survey



(b) Marine seismic survey

Outline

- Problem Statement

Outline

- Problem Statement
- Mathematical Result

Outline

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- Mathematical Result
- Computational Methodology

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- Computational Methodology
- Numerical Results

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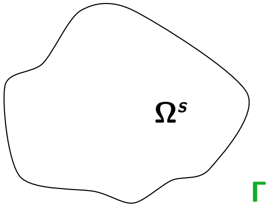
- Problem Statement
- Mathematical Result
- Computational Methodology
- Numerical Results
- Conclusions and Perspectives

Objective

Recovery of shape and material parameters of

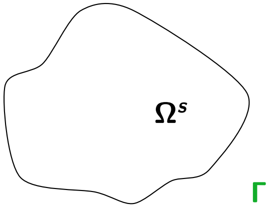
an elastic homogeneous object in a fluid

Problem statement



Problem statement

$$\Omega^f = \mathbb{R}^n \setminus \overline{\Omega}^s$$



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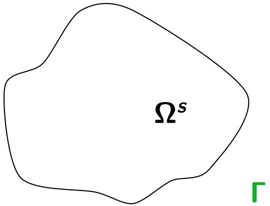
- Source



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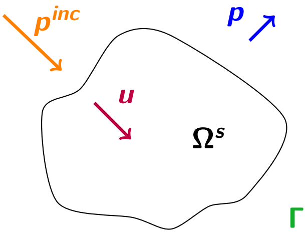
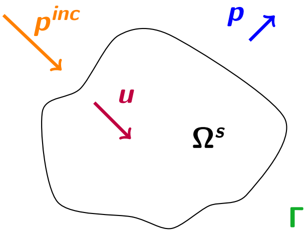
- Source



Problem statement

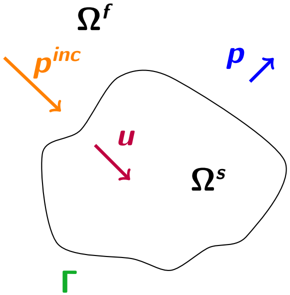
$$\Omega^f = \mathbb{R}^n \setminus \overline{\Omega}^s$$

- Source
- Receiver



Problem statement(2)

THE DIRECT ELASTO-ACOUSTIC SCATTERING PROBLEM



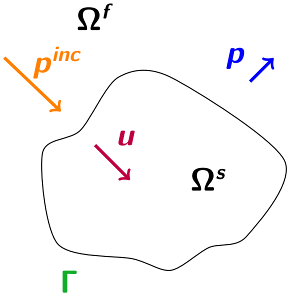
Problem statement(2)

THE DIRECT ELASTO-ACOUSTIC SCATTERING PROBLEM

$$\nabla \cdot \sigma(\boldsymbol{u}) \quad + \quad \omega^2 \rho_s \boldsymbol{u} = \boldsymbol{0} \qquad \text{in } \Omega^s$$

$$\Delta p \quad + \quad k^2 p = 0 \qquad \text{in } \Omega^f$$

$$k = \left(\frac{\omega}{c_f}\right)$$



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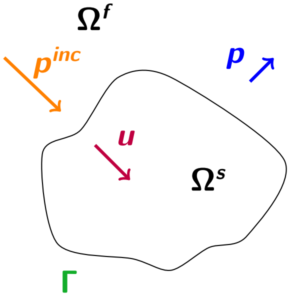

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$$\sigma(u) n = -p n - p^{inc} \quad \text{on } \Gamma$$

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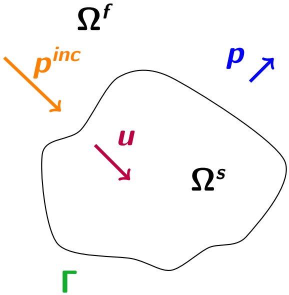

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$$\lim_{x \to +\infty} r\left(\frac{\partial p}{\partial r} - ikp\right) = 0 \quad (r = \| x \|_2)$$

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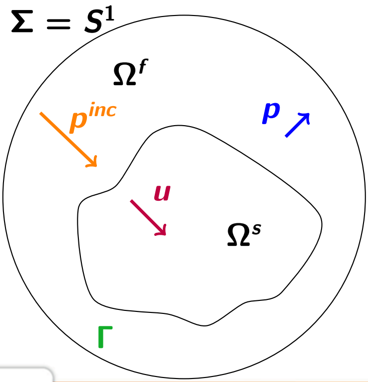

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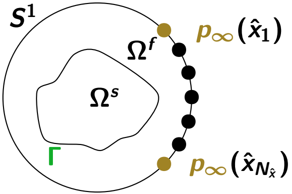
Problem statement(3)

THE FAR FIELD PATTERN



$$p(x) = \frac{e^{ikr}}{\sqrt{r}} \left(p_\infty \left(\frac{x}{r} \right) + \mathcal{O} \left(\frac{x}{r} \right) \right)$$

$$r = \| x \|_2 \rightarrow \infty$$

$$p_\infty(\hat{x}) = \frac{e^{i\pi \backslash 4}}{\sqrt{8\pi k}} \int_\Gamma \left(e^{ik\hat{x}\cdot y} \frac{\partial p}{\partial n} - \frac{\partial e^{ik\hat{x}\cdot y}}{\partial n} p(y) \right) d\Gamma$$

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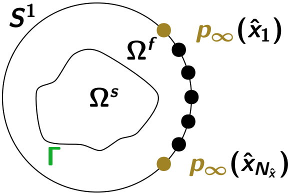

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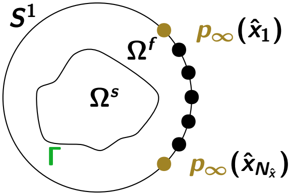

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David Colton and Rainer Kress
Inverse Acoustic and Electromagnetic Scattering Theory
Applied Mathematical Sciences

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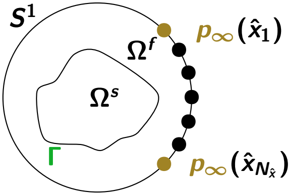

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The FFP intensity

$$p^*(\hat{x}) = \bar{p}_\infty(\hat{x}) \cdot p_\infty(\hat{x})$$

Problem statement(4)

- **Direct mapping**

$$\boldsymbol{\Upsilon}, \boldsymbol{\Gamma} \longmapsto \boldsymbol{F}(\boldsymbol{\Upsilon}, \boldsymbol{\Gamma}) = \boldsymbol{p}_\infty$$

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$$(\mathrm{IP}) = \begin{cases} \text{Given FFP measurements } \tilde{\boldsymbol{p}}_\infty(\boldsymbol{x}_j), \\ \text{Find } \boldsymbol{\Upsilon}, \boldsymbol{\Gamma}, \text{ such that} \\ \boldsymbol{F}(\boldsymbol{\Upsilon}, \boldsymbol{\Gamma})(\boldsymbol{x}_j) = \tilde{\boldsymbol{p}}_\infty(\boldsymbol{x}_j); \quad \boldsymbol{j} = 1, \ldots, \boldsymbol{M} \end{cases}$$

Problem statement(4)

- **Direct mapping**

$$\Upsilon, \Gamma \longmapsto F(\Upsilon, \Gamma) = p_\infty$$

$$\Upsilon, \Gamma \longmapsto A(\Upsilon, \Gamma) = \bar{F}(\Upsilon, \Gamma) \cdot F(\Upsilon, \Gamma) = \bar{p}_\infty p_\infty = p^*$$

- **Inverse Problem**

$$(\mathrm{IP}) = \begin{cases} \text{Given FFP measurements } \tilde{p}_\infty(x_j), \\ \text{Find } \Upsilon, \Gamma, \text{ such that} \\ F(\Upsilon, \Gamma)(x_j) = \tilde{p}_\infty(x_j); \quad j = 1, \ldots, M \end{cases}$$

Problem statement(4)

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$\Longrightarrow$ **Ill-Posed & Nonlinear problem**

DIFFERENT INVERSE PROBLEMS

- ISOTROPIC CASE

$$(\text{IP-1}) \begin{cases} \text{Find } \lambda, \mu, \Gamma, \text{ such that} \\ \\ A(\lambda, \mu, \Gamma)(x_j) = \bar{p}_\infty(x_j) p_\infty(x_j) = \tilde{p}^*(x_j); \quad j = 1, ..., M \end{cases}$$

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$$(\text{IP-2}) \begin{cases} \text{Find } \rho, \Gamma, \text{ such that} \\ A(\rho, \Gamma)(x_j) = \bar{p}_\infty(x_j) p_\infty(x_j) = \tilde{p}^*(x_j); \quad j = 1, ..., M \\ F(x_{ref}, y_{ref})(x_j) = \tilde{p}_\infty(x_j); \quad j = 1, ..., M \end{cases}$$

DIFFERENT INVERSE PROBLEMS

- ISOTROPIC CASE

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- ANISOTROPIC CASE

$$(\text{IP-3}) \begin{cases} \text{Find } \rho, V_p, V_s, \epsilon, \delta, \Gamma, \text{ such that} \\ \\ A(\rho, V_p, V_s, \epsilon, \delta, \Gamma)(x_j) = \bar{p}_\infty(x_j) p_\infty(x_j) = \tilde{p}^*(x_j); \quad j = 1, ..., M \end{cases}$$

Problem statement(4)

THE NONLINEARITY ISSUE

Find the Lamé coefficients λ, μ and the shape Γ such that

$$\min_{\lambda,\mu,\Gamma} \frac{1}{2} || \sum_{j=1}^{N_x} A(\lambda, \mu, \Gamma)(\hat{x}_j) - \overline{\tilde{p}_\infty}(\hat{x}_j)\, \tilde{p}_\infty(\hat{x}_j) ||_2^2$$

THE NONLINEARITY ISSUE

Define

$$J\left(\lambda,\mu,\Gamma\right) = \frac{1}{2}\|\sum_{j=1}^{N_x} A\left(\lambda,\mu,\Gamma\right)\left(\hat{x}_j\right) - \overline{\tilde{p}_\infty}\left(\hat{x}_j\right)\tilde{p}_\infty\left(\hat{x}_j\right)\|^2$$

Find the Lamé coefficients λ, μ and the shape Γ such that

$$J(\lambda,\mu,\Gamma) = 0$$

Problem statement(5)

THE NONLINEARITY ISSUE

The Newton Iteration

$$\left(A'_{\lambda}(\hat{x}_j), A'_{\mu}(\hat{x}_j), A'_{\Gamma}(\hat{x}_j)\right) \cdot \begin{pmatrix} \delta_{\lambda} \\ \delta_{\mu} \\ \delta_{\Gamma} \end{pmatrix} = A\left(\lambda^n, \mu^n, \Gamma^n\right)(\hat{x}_j) - \overline{\tilde{p}_{\infty}}\tilde{p}_{\infty}(\hat{x}_j)$$

$$j = 1, \ldots, N_x$$

Update the parameters

$$\left(\lambda^{n+1}, \mu^{n+1}, \Gamma^{n+1}\right) = \left(\lambda^n, \mu^n, \Gamma^n\right) + \left(\delta_{\lambda}, \delta_{\mu}, \delta_{\Gamma}\right)$$

Problem statement(5)

The Newton Iteration

$$B \cdot \delta\delta = f$$

$$B = \begin{pmatrix} A'^T_\lambda(\hat{x}_j) \\ A'^T_\mu(\hat{x}_j) \\ A'^T_\Gamma(\hat{x}_j) \end{pmatrix} \left(A'_\lambda(\hat{x}_j), A'_\mu(\hat{x}_j), A'_\Gamma(\hat{x}_j)\right) \qquad j = 1, \ldots, N_x$$

$$f = \begin{pmatrix} A'^T_\lambda(\hat{x}_j) \\ A'^T_\mu(\hat{x}_j) \\ A'^T_\Gamma(\hat{x}_j) \end{pmatrix} \left(A(\lambda^n, \mu^n, \Gamma^n)(\hat{x}_j) - \overline{\tilde{p}_\infty \tilde{p}_\infty}(\hat{x}_j)\right) \qquad j = 1, \ldots, N_x$$

Update the parameters

$$(\lambda^{n+1}, \mu^{n+1}, \Gamma^{n+1}) = (\lambda^n, \mu^n, \Gamma^n) + (\delta_\lambda, \delta_\mu, \delta_\Gamma)$$

Problem statement(5)

The Newton Iteration

$$B \cdot \delta \quad + \quad \alpha \cdot \delta = (B + \alpha \cdot I)\, \delta = f$$

$$B = \begin{pmatrix} A_{\lambda}'^{T}(\hat{x}_j) \\ A_{\mu}'^{T}(\hat{x}_j) \\ A_{\Gamma}'^{T}(\hat{x}_j) \end{pmatrix} \left(A_{\lambda}'(\hat{x}_j), A_{\mu}'(\hat{x}_j), A_{\Gamma}'(\hat{x}_j)\right) \qquad j = 1, \ldots, N_x$$

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Update the parameters

$$(\lambda^{n+1}, \mu^{n+1}, \Gamma^{n+1}) = (\lambda^n, \mu^n, \Gamma^n) + (\delta_{\lambda}, \delta_{\mu}, \delta_{\Gamma})$$

Mathematical Result

$$(\boldsymbol{p}, \boldsymbol{u}) \in \boldsymbol{\Omega}^{f} \times \boldsymbol{\Omega}^{s}$$

- Lamé coefficients:
  - $\lambda \rightarrow (\boldsymbol{p}', \boldsymbol{u}') = (\partial_{\lambda}\boldsymbol{p}, \partial_{\lambda}\boldsymbol{u})$
  - $\mu \rightarrow (\boldsymbol{p}', \boldsymbol{u}') = (\partial_{\mu}\boldsymbol{p}, \partial_{\mu}\boldsymbol{u})$
- Shape parameters: $s_1, \ldots, s_N \rightarrow \Gamma = \Gamma(s_1, \ldots, s_N)$
  - $s_i \rightarrow (\boldsymbol{p}', \boldsymbol{u}') = (\frac{\partial p}{\partial \Gamma}\frac{\partial \Gamma}{\partial s_i}, \frac{\partial u}{\partial \Gamma}\frac{\partial \Gamma}{\partial s_i}) \qquad i = 1, \ldots, N$

Mathematical Result

Characterization of the Fréchet derivative with respect to shape parameters and location

$$
\begin{aligned}
\nabla\cdot\sigma(u') + \omega^2\rho_s u' &= 0 && \text{in } \Omega^s\\
\Delta p' + k^2 p' &= 0 && \text{in } \Omega^f\\
\sigma(u')\cdot\nu + p'\cdot\nu &= F_j(u,p,h_j) && \text{on } \Gamma\\
\omega^2\rho_s u'\cdot\nu - \partial_\nu p' &= G_j(u,p,h_j) && \text{on } \Gamma\\
\partial_\nu p' - ikp' &= 0 && \text{on } \Sigma
\end{aligned}
$$

where, for $p^T = p + p^{inc}$ the function F and G are given by

$$
\begin{aligned}
F_j(u,p,h_j) &= -\,h_j^t\nabla\sigma(u)\nu - \nabla p^T\cdot h_j\nu + \sigma(u)[h_j']^t\nu + p^T[h_j']^t\nu\\
G_j(u,p,h_j) &= -\,(\omega^2\rho_f\nabla u - \nabla(\nabla p^T))h_j\cdot\nu + (\omega^2\rho_f u\nu - \nabla p^T)\cdot[h_j']^t\nu
\end{aligned}
$$

H. Barucq, R. Djellouli, and E. Estecahandy
Efficient DG-like formulation equipped with curved boundary edges for solving elasto-acoustic scattering problems

Mathematical Result

Characterization of the Fréchet derivative with respect to the material parameters

$$
\begin{aligned}
\nabla \cdot \sigma(u') + \omega^2 \rho_s u' &= -\nabla \cdot (C' \varepsilon(u)) && \text{in } \Omega^s \\
\Delta p' + k^2 p' &= 0 && \text{in } \Omega^f \\
\sigma(u') \cdot \nu + p' \cdot \nu &= C' \varepsilon(u) \cdot \nu && \text{on } \Gamma \\
\omega^2 \rho_s u' \cdot \nu + \partial_\nu p' &= 0 && \text{on } \Gamma \\
\partial_\nu p' - ikp' &= 0 && \text{on } \Sigma
\end{aligned}
$$

Mathematical Result

Characterization of the Fréchet derivative with respect to the material parameters

$$
\begin{aligned}
\nabla\cdot\sigma(\boldsymbol{u'})+\omega^2\rho_s\boldsymbol{u'} &= -\,\omega^2\boldsymbol{u}-\nabla\cdot(\boldsymbol{C'}\varepsilon(\boldsymbol{u})) && \text{in } \boldsymbol{\Omega^s}\\
\boldsymbol{\Delta p'}+\boldsymbol{k}^2\boldsymbol{p'} &= 0 && \text{in } \boldsymbol{\Omega^f}\\
\sigma(\boldsymbol{u'})\cdot\nu+\boldsymbol{p'}\cdot\nu &= \boldsymbol{C'}\varepsilon(\boldsymbol{u})\cdot\nu && \text{on } \boldsymbol{\Gamma}\\
\omega^2\rho_s\boldsymbol{u'}\cdot\nu+\partial_\nu\boldsymbol{p'} &= 0 && \text{on } \boldsymbol{\Gamma}\\
\partial_\nu\boldsymbol{p'}-\boldsymbol{ik}\boldsymbol{p'} &= 0 && \text{on } \boldsymbol{\Sigma}
\end{aligned}
$$

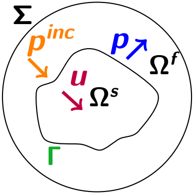
Mathematical Result

Characterization of the Fréchet derivative with respect to the density

$$
\begin{aligned}
\nabla \cdot \sigma(\boldsymbol{u}') + \omega^2 \rho_s \boldsymbol{u}' &= -\,\omega^2 \boldsymbol{u} && \text{in } \Omega^s \\
\Delta \boldsymbol{p}' + \boldsymbol{k}^2 \boldsymbol{p}' &= 0 && \text{in } \Omega^f \\
\sigma(\boldsymbol{u}') \cdot \nu + \boldsymbol{p}' \cdot \nu &= 0 && \text{on } \Gamma \\
\omega^2 \rho_s \boldsymbol{u}' \cdot \nu + \partial_\nu \boldsymbol{p}' &= 0 && \text{on } \Gamma \\
\partial_\nu \boldsymbol{p}' - \boldsymbol{ik}\boldsymbol{p}' &= 0 && \text{on } \Sigma
\end{aligned}
$$

Solution methodology
for the direct elasto-acoustic scattering problem

Main features



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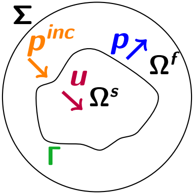
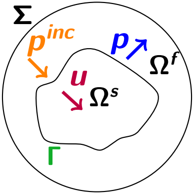


Main features

- A Discontinuous Galerkin method with interior penalty (IPDG)

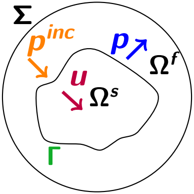
Solution methodology
for the direct elasto-acoustic scattering problem



Main features

- A Discontinuous Galerkin method with interior penalty (IPDG)
- Higher-order elements

Solution methodology for the direct elasto-acoustic scattering problem



Main features

- A Discontinuous Galerkin method with interior penalty (IPDG)
- Higher-order elements
- Curved edges on the boundaries $\boldsymbol{\Gamma}$ and $\boldsymbol{\Sigma}$

Definitions of errors

- Relative Residual(RR) $= \frac{\|\overline{p_\infty} \cdot p_\infty(\lambda,\mu,\Gamma) - \overline{p_\infty} \cdot p_\infty(\lambda^n,\mu^n,\Gamma^n)\|}{\|\overline{p_\infty} \cdot p_\infty(\lambda,\mu,\Gamma)\|}$

Definitions of errors

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- Relative Error of Shape Param.(Err$_{shap}$)$= \frac{\left(\sum_{i=1}^{k} \|s_i - s_i{}^n\|^2\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{k} \|s_i\|^2\right)^{\frac{1}{2}}}$

Definitions of errors

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- Relative Error of Shape Param.(Err_{shap})$= \frac{\left(\sum_{i=1}^{k}\|s_i-s_i{}^n\|^2\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{k}\|s_i\|^2\right)^{\frac{1}{2}}}$
- Relative Error of Material Param.(Err_{mat})$= \frac{\left(\|\lambda-\lambda^n\|^2+\|\mu-\mu^n\|^2\right)^{\frac{1}{2}}}{\left(\|\lambda\|^2+\|\mu\|^2\right)^{\frac{1}{2}}}$

Definitions of errors

Individual Relaltive Errors of Material parameters

- $\boldsymbol{Err_\rho = \frac{\|\rho - \rho^n\|}{\|\rho\|}}$

Definitions of errors

Individual Relaltive Errors of Material parameters

- $\boldsymbol{Err_{\rho} = \frac{\|\rho - \rho^n\|}{\|\rho\|}}$
- $\boldsymbol{Err_{V_p} = \frac{\|V_p - V_p^n\|}{\|V_p\|}}$

Definitions of errors

Individual Relaltive Errors of Material parameters

- $Err_{\rho} = \frac{\|\rho - \rho^n\|}{\|\rho\|}$
- $Err_{V_p} = \frac{\|V_p - V_p^n\|}{\|V_p\|}$
- $Err_{V_s} = \frac{\|V_s - V_s^n\|}{\|V_s\|}$

Definitions of errors

Individual Relaltive Errors of Material parameters

- $Err_{\rho} = \frac{||\rho - \rho^n||}{||\rho||}$
- $Err_{V_p} = \frac{||V_p - V_p^n||}{||V_p||}$
- $Err_{V_s} = \frac{||V_s - V_s^n||}{||V_s||}$
- $Err_{\epsilon} = \frac{||\epsilon - \epsilon^n||}{||\epsilon||}$

Definitions of errors

Individual Relaltive Errors of Material parameters

- $Err_{\rho} = \frac{||\rho - \rho^n||}{||\rho||}$
- $Err_{V_p} = \frac{||V_p - V_p^n||}{||V_p||}$
- $Err_{V_s} = \frac{||V_s - V_s^n||}{||V_s||}$
- $Err_{\epsilon} = \frac{||\epsilon - \epsilon^n||}{||\epsilon||}$
- $Err_{\delta} = \frac{||\delta - \delta^n||}{||\delta||}$

Elliptic coordinate system

s_1, s_2

Elliptic coordinate system

$$s_1, s_2 \to \Gamma = \{(x_{ref}, y_{ref}) + (s_1 \cos\theta_j, s_2 \sin\theta_j)\,, \quad \theta_j \in [0, 2\pi)\}$$

Polygonal parametrization

$s_1, s_2, \ldots, s_N$

Polygonal parametrization

$$s_1, s_2, \dots, s_N \to X_j = (x_{ref}, y_{ref}) + s_j \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix}, \; j = 1, \dots, N,$$

$$T_j(t) = \begin{pmatrix} t & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_j \\ X_{j+1} \end{pmatrix}$$

with $X_{N+1} = X_1, \qquad t \in [0,1]$

$$\Gamma = \{ T_j(t), \quad t \in [0,1], j = 1, \dots, N \}$$

B-spline representation

$s_1, s_2, \ldots, s_N$

B-spline representation

$$s_1, s_2, ..., s_N \to \boldsymbol{X_j} = (x_{ref}, y_{ref}) + s_j \begin{pmatrix} cos\theta_j \\ sin\theta_j \end{pmatrix}, \; j = 1, ..., N,$$

$$T_j(t) = \frac{1}{2} \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{X_{j-1}} \\ \boldsymbol{X_j} \\ \boldsymbol{X_{j+1}} \end{pmatrix}$$

with $\boldsymbol{X_{N+1}} = \boldsymbol{X_1}$ and $\boldsymbol{X_0} = \boldsymbol{X_N}$.

$$\Gamma = \{ T_j(t), \quad t \in [0, 1], \quad j = 1, ..., N \}$$

Fourier Parametrization

$$\Gamma = \{(x_{ref}, y_{ref}) + r(\theta)\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi)\}$$

where $r^M(\theta) = s_1 + \sum_{k=1}^{M} s_{2k} cos(k\theta) + s_{2k+1} sin(k\theta)$

$$\Gamma = \{(x_{ref}, y_{ref}) + \sum_{k=1}^{M} s_j \phi_j(\theta)\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi)\}$$

where $\phi_1 = 1, \phi_{2k} = \cos(k\theta), \phi_{2k+1} = \sin(k\theta)$ for $k = 1, ..., M$

Isotropic cases

Steel

$$\lambda = 9.695 \cdot 10^9 \quad N/m^2 \qquad \mu = 7.617 \cdot 10^9 N/m^2$$

$$\mathrm{Err}_{mat}{}^0 = 43.59\%$$
$$\mathrm{Err}\lambda^0 = 48.42\%$$
$$\mathrm{Err}\mu^0 = 34.35\%$$

Cases depending on shape parametrization

- Elliptic coordinate system

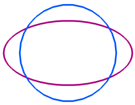


Figure : Ellipse

- Polygonal shaped objects

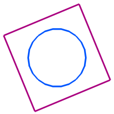


Figure : Square

- B-spline representation

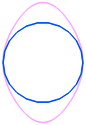
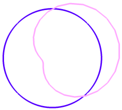
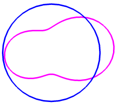


Figure : Rugby

- Fourier Parametrization



(a) Potato



(b) Peanut

Computational Methodology

Algorithm 1

- Synthetic data, $p_{\infty,meas}$
- Initialization
  $\omega^0, \alpha^0 \quad \&\& \quad \lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma(s_1{}^0, ..., s_r{}^0)$
- **WHILE** Relative Residual (RR)$>$ $\mathbf{0.01}$ $\quad \&\& \quad$ $\boldsymbol{n}$ $<$MaxNumberIter

Computational Methodology

Algorithm 1

- Synthetic data, $p_{\infty,meas}$
- Initialization
  $\omega^0, \alpha^0 \quad \&\& \quad \lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma(s_1{}^0, ..., s_r{}^0)$
- **WHILE** Relative Residual (RR)$>$ **0.01** **&&** n $<$MaxNumberIter
  - stagnation=100
  - **WHILE** stagnation$>$ **0.01**

Computational Methodology

Algorithm 1

- Synthetic data, $p_{\infty,meas}$
- Initialization
  $\omega^0, \alpha^0 \quad \&\& \quad \lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma({s_1}^0, \ldots, {s_r}^0)$
- **WHILE** Relative Residual (RR)$>$ **0.01** $\quad \&\& \quad n <$MaxNumberIter
  - stagnation=100
  - **WHILE** stagnation$>$ **0.01**
    - Far Field Pattern $p_\infty(\lambda^n, \mu^n, \Gamma^n)$
    - Fréchet derivative, $\frac{\partial p_\infty}{\partial \lambda}, \frac{\partial p_\infty}{\partial \mu}, \frac{\partial p_\infty}{\partial s_i} \quad i = 1, \ldots, r$
    - Parameters update , $\lambda^{n+1}, \mu^{n+1}, {s_1}^{n+1}, \ldots, {s_r}^{n+1}$
    - stagnation $= \frac{p_\infty^n - p_\infty^{n+1}}{p_\infty^n} \cdot 100$
    - n=n+1

Computational Methodology

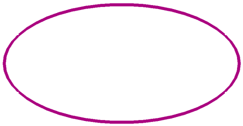
Algorithm 1

- Synthetic data, $p_{\infty,meas}$
- Initialization
  ω^0, α^0 && $\lambda^0, \mu^0, \Gamma^0$: $\Gamma^0 = \Gamma({s_1}^0, ..., {s_r}^0)$
- **WHILE** Relative Residual (RR)$>$ **0.01** && n <MaxNumberIter
  - stagnation=100
  - **WHILE** stagnation$>$ **0.01**
    - Far Field Pattern $p_\infty(\lambda^n, \mu^n, \Gamma^n)$
    - Fréchet derivative, $\frac{\partial p_\infty}{\partial \lambda}, \frac{\partial p_\infty}{\partial \mu}, \frac{\partial p_\infty}{\partial s_i}$ $i = 1, ..., r$
    - Parameters update , $\lambda^{n+1}, \mu^{n+1}, {s_1}^{n+1}, ..., {s_r}^{n+1}$
    - stagnation $= \frac{p_\infty^n - p_\infty^{n+1}}{p_\infty^n} \cdot 100$
    - n=n+1
  - Increase the frequency (*) $p_{\infty,meas}$

Elliptic coordinate system

Ellipse,Steel media, Algorithm1, Noise Level$=0\%$

$$s_1, s_2 \to \Gamma = \{(x_{ref}, y_{ref}) + (s_1 \cos\theta, s_2 \sin\theta)\,, \quad \theta \in [0, 2\pi)\}$$

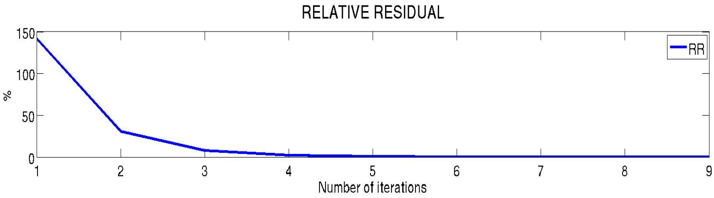


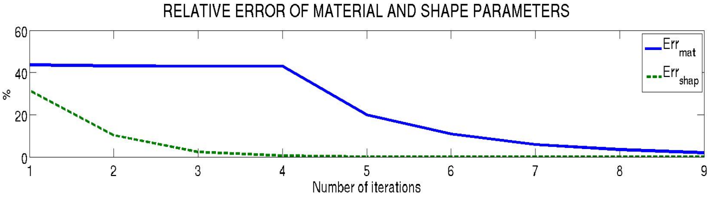
	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

$k_f = 267, \alpha = 0.5$ (after 4iter $\alpha = 5 \cdot 10^{-6}$)

Elliptic coordinate system

Ellipse,Steel media, Algorithm1, Noise Level$= 0\%$

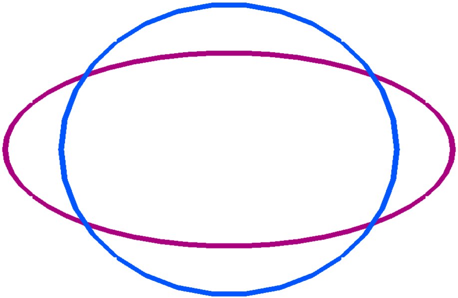




Elliptic coordinate system

Ellipse,Steel media, Algorithm1, Noise Level$= 0\%$

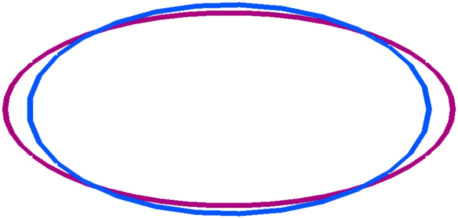
$\#Iter = 0$



Elliptic coordinate system

Ellipse,Steel media, Algorithm1, Noise Level$= 0\%$

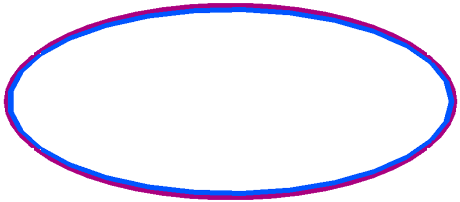
$\#Iter = 1$



Elliptic coordinate system

Ellipse,Steel media, Algorithm1, Noise Level$= 0\%$

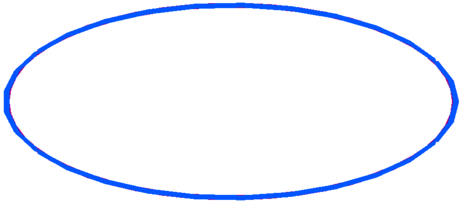
$\#Iter = 2$



Elliptic coordinate system

Ellipse,Steel media, Algorithm1, Noise Level$= 0\%$

$\#Iter = 3$



Polygonal parametrization

$s_1, s_2, \ldots, s_N$

Polygonal parametrization

$$s_1, s_2, \ldots, s_N \to X_j = (x_{ref}, y_{ref}) + s_j \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix}, \; j = 1, \ldots, N,$$

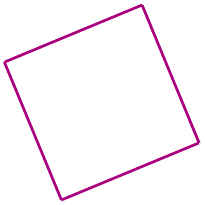
$$T_j(t) = \begin{pmatrix} t & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_j \\ X_{j+1} \end{pmatrix}$$

with $X_{N+1} = X_1, \qquad t \in [0, 1]$

$$\Gamma = \{ T_j(t), \quad t \in [0, 1], j = 1, \ldots, N \}$$

Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level$= 0\%$



	Target	In.Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

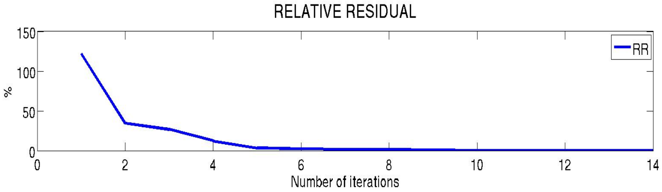
Polygonal parametrization

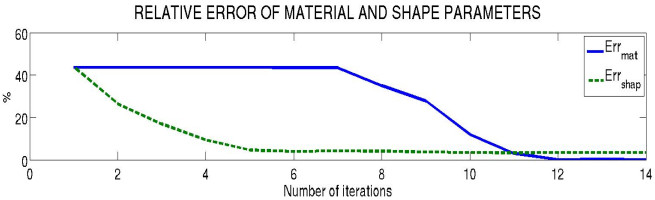
Square, Steel media, Algorithm1, Noise Level= 0%

# Iter	k_f	TRR(%)	Err_{mat}(%)	Err_{shap}(%)	α
1	66.67	121.35	43.60	39.91	2
2	100	33.13	43.61	22.76	2
3	133.33	23.91	43.61	12.36	2
4	166.67	13.19	43.60	5.92	2
5	200	3.00	43.57	2.21	2
6	233.33	1.34	43.52	1.20	2
7	266.67	1.88	43.43	1.31	$2 \cdot 10^{-6}$
8	300	1.02	15.44	0.74	$2 \cdot 10^{-6}$
9	333.33	0.27	10.82	0.22	$2 \cdot 10^{-6}$
10	366.67	0.10	3.84	0.09	$2 \cdot 10^{-6}$
11	400	0.02	0.13	0.01	$2 \cdot 10^{-6}$

Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level$= 0\%$

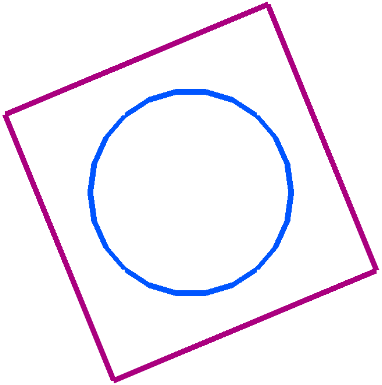




Polygonal parametrization

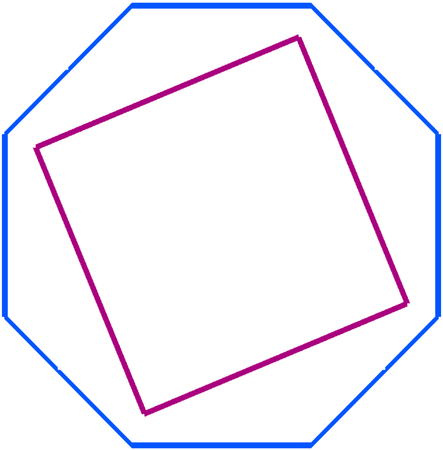
Square, Steel media, Algorithm1, Noise Level $= 0\%$

$\#$**Iter=0**



Polygonal parametrization

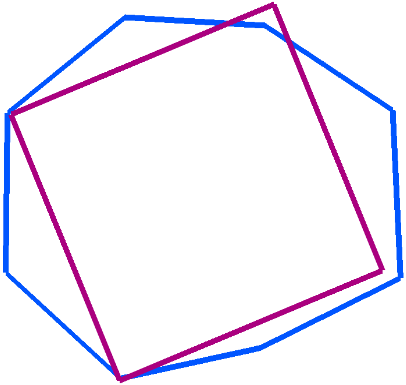
Square, Steel media, Algorithm1, Noise Level= 0%
#Iter=1



Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level$= 0\%$

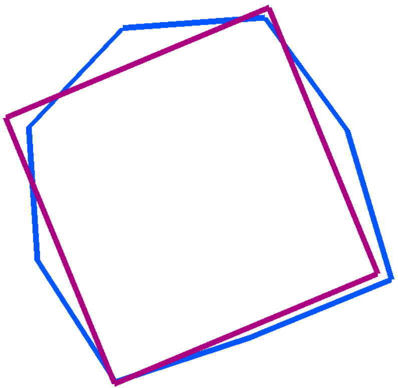
$\#$Iter=2



Polygonal parametrization

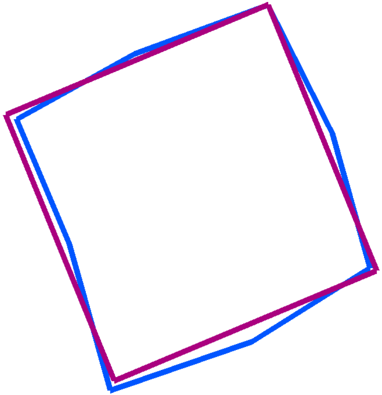
Square, Steel media, Algorithm1, Noise Level$= 0\%$

$\#$**Iter=3**



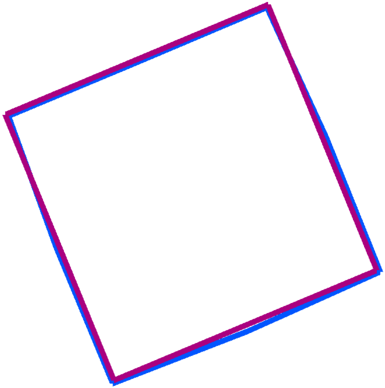
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level $= 0\%$
$\#$**Iter=4**



Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level= 0%
#**Iter=5**



B-spline representation

$s_1, s_2, \ldots, s_N$

B-spline representation

$$s_1, s_2, \ldots, s_N \to \boldsymbol{X_j} = (x_{ref}, y_{ref}) + s_j \begin{pmatrix} cos\theta_j \\ sin\theta_j \end{pmatrix}, \; j = 1, \ldots, N,$$

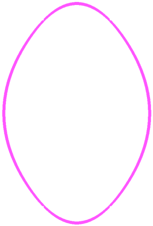
$$T_j(t) = \frac{1}{2} \begin{pmatrix} t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{X_{j-1}} \\ \boldsymbol{X_j} \\ \boldsymbol{X_{j+1}} \end{pmatrix}$$

with $\boldsymbol{X_{N+1}} = \boldsymbol{X_1}$ and $\boldsymbol{X_0} = \boldsymbol{X_N}$.

$$\Gamma = \{ T_j(t), \quad t \in [0, 1], \quad j = 1, \ldots, N\}$$

B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$



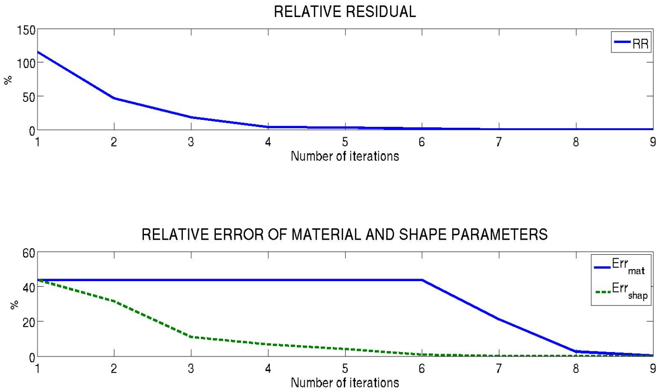
	Target	In.Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175

Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$

# Iter	k_f	TRR(%)	Err_{mat}(%)	Err_{shap}(%)	α
1	66.67	115.47	43.60	43.85	1
2	100	46.13	43.61	31.37	1
3	133.337	18.19	43.60	10.90	1
4	166.67	3.54	43.60	6.71	1
5	200	2.79	43.60	4.05	1
6	233.33	1.21	43.59	0.89	10^{-9}
7	266.67	0.29	21.04	0.06	10^{-9}
8	300	0.04	2.61	0.01	10^{-9}

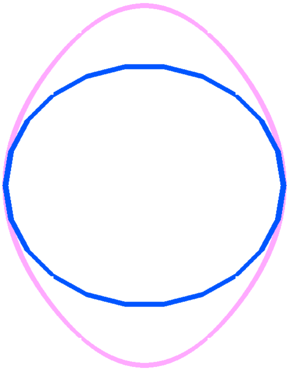
B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$



B-Spline representation

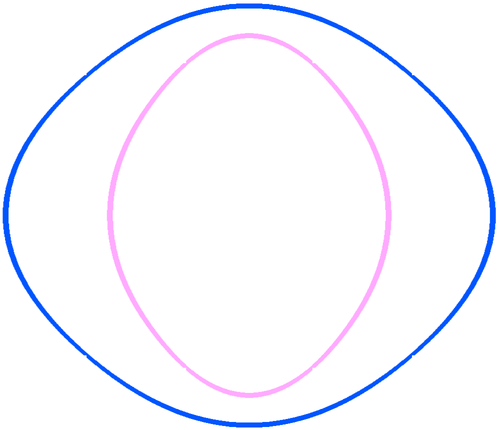
Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$
$\#$**Iter=0**



B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$

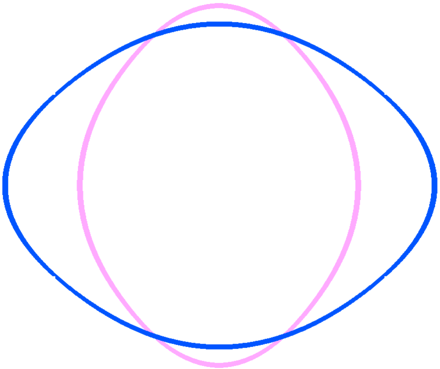
$\#$**Iter=1**



B-Spline representation

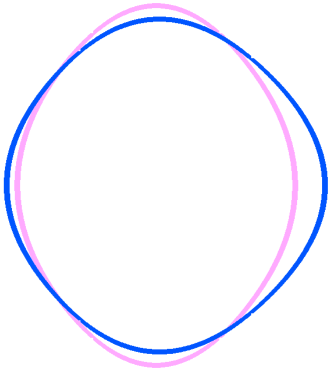
Rugby ball, Steel media, Algorithm1, Noise Level $= 0\%$

$\#$**Iter=2**



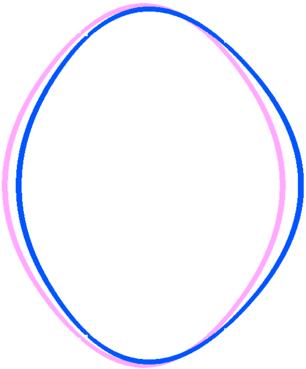
B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$ $\#$Iter=3



B-Spline representation

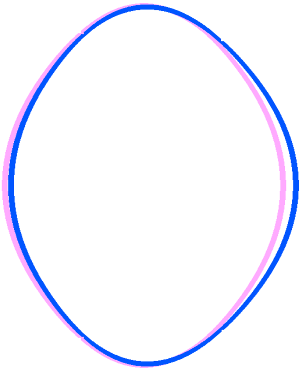
Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$
$\#$**Iter=4**



B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$

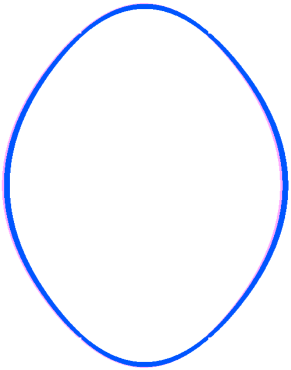
$\#$**Iter=5**



B-Spline representation

Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$

$\#$**Iter=6**



Fourier Parametrization

$$\Gamma = \{(x_{ref}, y_{ref}) + r(\theta)\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi)\}$$

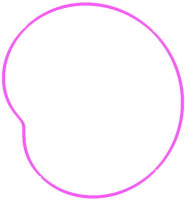
where $r^M(\theta) = s_1 + \sum_{k=1}^{M} s_{2k} cos(k\theta) + s_{2k+1} sin(k\theta)$

$$\Gamma = \{(x_{ref}, y_{ref}) + \sum_{k=1}^{M} s_j \phi_j(\theta)\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi)\}$$

where $\phi_1 = 1, \phi_{2k} = \cos(k\theta), \phi_{2k+1} = \sin(k\theta)$ for $k = 1, ..., M$

Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level$= 0\%$



	Target	In.Guess
s_1	0.01	0.0125
s_2	0.007	0
s_3	0.0025	0

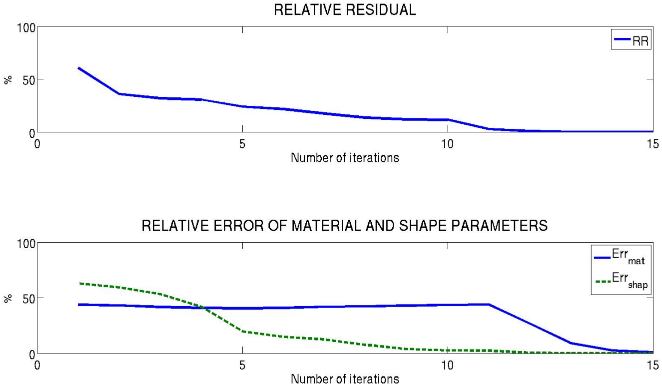
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level $= 0\%$

# Iter	k_f	TRR(%)	Err_{mat}(%)	Err_{shap}(%)	α
1	800	60.51	43.60	62.94	12
2	800	35.74	42.94	59.18	12
3	800	31.77	41.68	53.23	12
5	800	23.97	40.30	19.76	12
7	800	17.39	41.82	12.40	12
9	800	11.75	42.86	3.68	12
11	200	2.60	43.78	2.31	10^{-6}
12	200	0.81	26.55	0.28	10^{-6}
13	200	0.08	8.99	0.02	10^{-6}
14	200	0.01	2.46	0.02	10^{-6}
15	200	0.01	0.64	0.01	10^{-6}

Fourier Parametrization

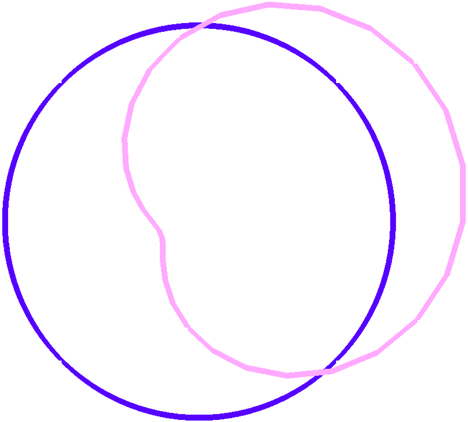
Potato, Steel media, Algorithm1, Noise Level$= 0\%$



Fourier Parametrization

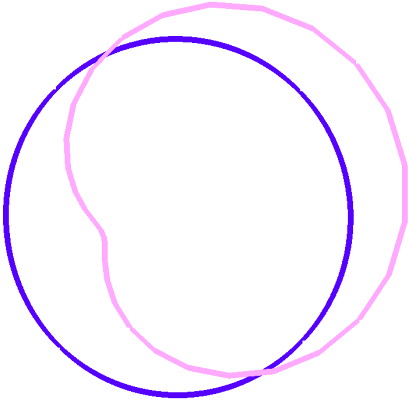
Potato, Steel media, Algorithm1, Noise Level$= 0\%$

#Iter=0



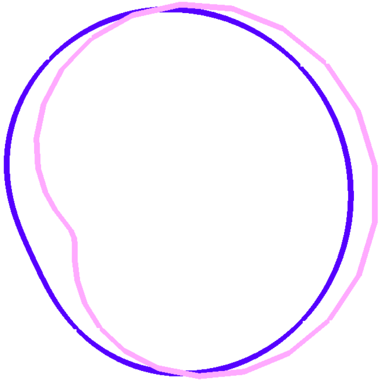
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level$= 0\%$
$\#$**Iter=3**



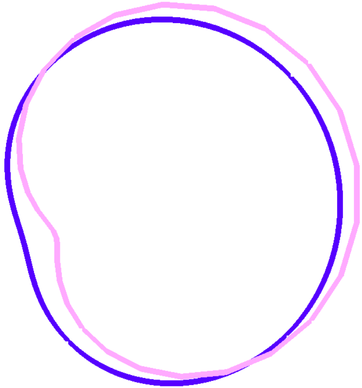
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%
#**Iter=4**



Fourier Parametrization

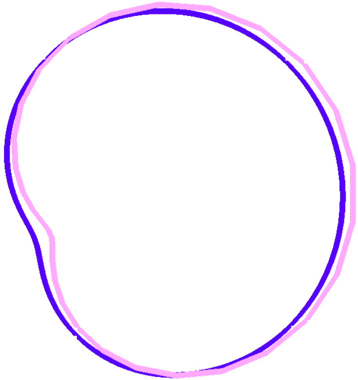
Potato, Steel media, Algorithm1, Noise Level$= 0\%$
$\#$**Iter=5**



Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level$= 0\%$

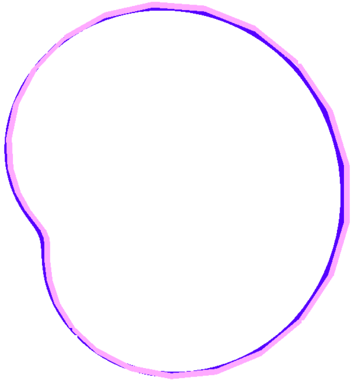
$\#$**Iter=7**



Fourier Parametrization

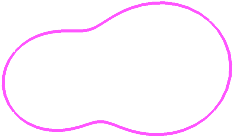
Potato, Steel media, Algorithm1, Noise Level$= 0\%$

$\#$**Iter=9**



Fourier Parametrization

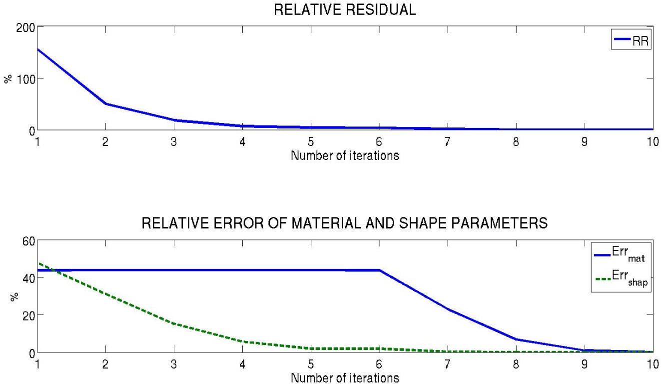
Peanut, Steel media, Algorithm1, Noise Level$= 0\%$



	Target	In.Guess
s_1	0.01	0.0125
s_2	0.002	0
s_3	0.0005	0
s_4	0.004	0
s_5	0.001	0

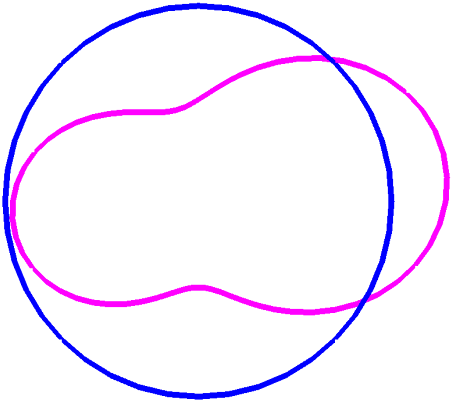
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level$= 0\%$



Fourier Parametrization

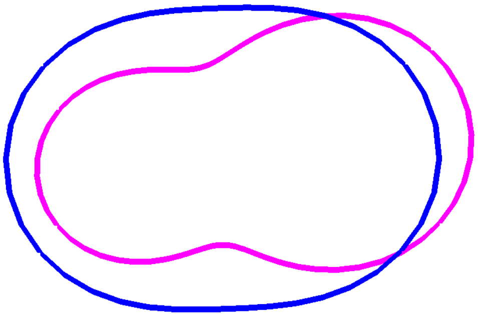
Peanut, Steel media, Algorithm1, Noise Level$= 0\%$
$\#$**Iter=0**



Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level$= 0\%$

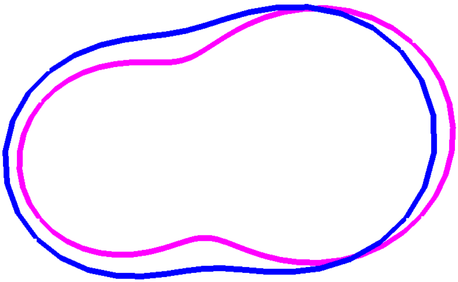
$\#$**Iter=1**



Fourier Parametrization

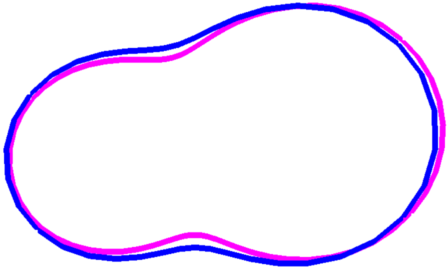
Peanut, Steel media, Algorithm1, Noise Level$= 0\%$

$\#$**Iter=2**



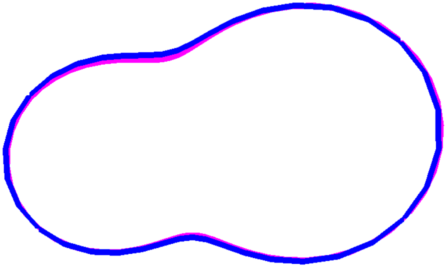
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level$= 0\%$
#Iter=3



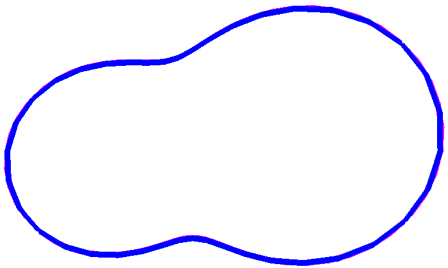
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level $= 0\%$
$\#$**Iter=4**



Fourier Parametrization

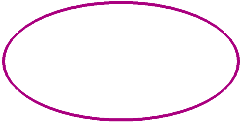
Peanut, Steel media, Algorithm1, Noise Level$= 0\%$
$\#$**Iter=6**



Elliptic coordinate system

Ellipse,Steel media, Algorithm1, Noise Level$= 0\%$

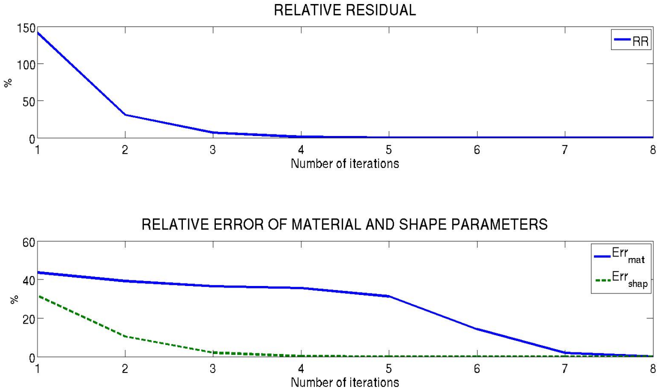
Various regularization parameters



	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level$= 0\%$



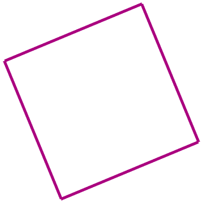
Elliptic coordinate system

Ellipse,Steel media, Algorithm1, Noise Level$= 0\%$

# Iter	k_f	TRR(%)	REMP(%)	RESP(%)	α_{mat}	α_{shape}
1	267	141.64	43.60	31.62	0.05	0.5
2	267	30.94	39.14	10.41	0.005	0.5
3	267	6.88	36.52	2.08	$5 \cdot 10^{-4}$	0.5
4	267	0.96	35.58	0.28	$5 \cdot 10^{-5}$	0.005
5	267	0.29	31.28	0.09	$5 \cdot 10^{-6}$	0.005
6	267	0.12	14.23	0.03	$5 \cdot 10^{-7}$	0.005
7	267	0.02	1.97	0.00	$5 \cdot 10^{-8}$	0.005
8	267	0.00	0.08	0.00	$5 \cdot 10^{-9}$	0.005

Polygonal parametrization

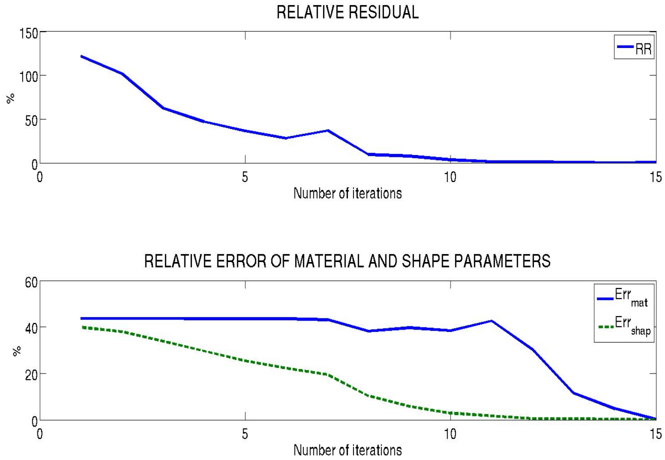
Square, Steel media, Algorithm1, Noise Level$= 0\%$
Various regularization parameters



	Target	In.Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level$= 0\%$



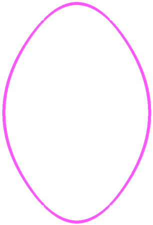
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level$= 0\%$

# Iter	k_f	TRR(%)	REMP(%)	RESP(%)	α_{mat}	α_{shape}
1	67	121.92	43.60	39.91	200	200
3	133	62.34	43.59	33.79	200	200
5	200	36.36	43.55	25.33	20	200
7	267	36.78	43.04	19.46	0.2	200
9	333	7.58	39.64	5.77	0.002	200
11	400	1.36	42.62	1.70	$2 \cdot 10^{-5}$	200
12	433	1.34	30.20	0.56	$2 \cdot 10^{-6}$	200
13	467	0.55	11.40	0.45	$2 \cdot 10^{-7}$	200
14	500	0.24	4.76	0.22	$2 \cdot 10^{-8}$	200
15	533	0.40	0.15	0.10	$2 \cdot 10^{-9}$	200

B-spline representation

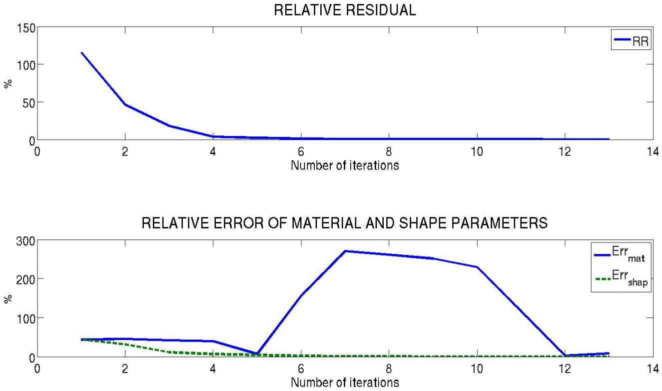
Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$
Various regularization parameters



	Target	In.Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175

B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$



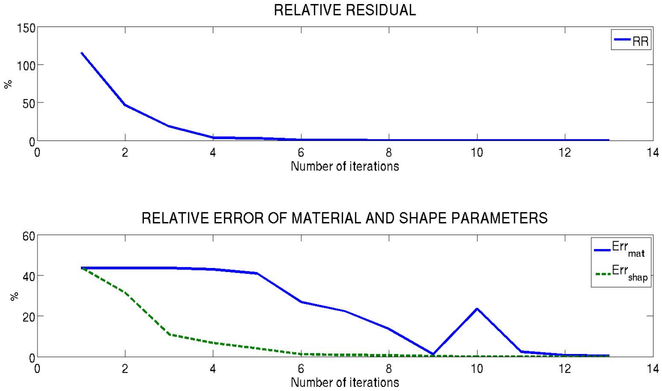
B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$

# Iter	k_f	TRR(%)	REMP(%)	RESP(%)	α_{mat}	α_{shape}
1	67	115.47	43.60	43.85	0.01	1
2	100	46.15	45.09	31.38	0.001	1
3	133	18.20	41.67	10.97	$2 \cdot 10^{-4}$	1
4	167	3.67	39.45	6.75	$2 \cdot 10^{-5}$	1
5	200	2.36	6.83	4.18	$2 \cdot 10^{-6}$	1
6	233	1.29	155.33	1.77	$2 \cdot 10^{-6}$	1
7	267	0.51	270.43	1.12	$2 \cdot 10^{-6}$	1
9	333	0.42	251.06	0.37	$2 \cdot 10^{-6}$	1
11	400	0.30	116.71	0.21	$2 \cdot 10^{-6}$	1
12	433	0.29	2.50	0.23	$2 \cdot 10^{-6}$	1

B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$



B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level$= 0\%$

# Iter	k_f	TRR(%)	REMP(%)	RESP(%)	α_{mat}	α_{shape}
1	67	115.47	43.60	43.85	1	1
2	100	46.13	43.61	31.37	0.1	1
3	133	18.19	43.56	10.90	0.01	1
4	167	3.55	42.91	6.71	10^{-3}	1
5	200	2.70	40.91	4.05	10^{-4}	1
6	233	0.74	26.80	1.19	10^{-5}	1
7	267	0.38	22.34	0.78	10^{-6}	1
8	300	0.26	13.54	0.58	10^{-7}	1
9	333	0.19	1.10	0.29	10^{-8}	1
10	367	0.08	23.52	0.05	10^{-9}	1
11	400	0.03	2.40	0.02	10^{-10}	1
12	433	0.01	0.73	0.01	10^{-11}	1

Computational Methodology

Algorithm 2

- Initialization $p_{\infty,meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma(s_1{}^0, ..., s_r{}^0)$
- **WHILE** Relative Residual (RR)$> 0.01 \quad \&\& \quad n <$MaxNumberIter

Computational Methodology

Algorithm 2

- Initialization $p_{\infty,meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma({s_1}^0, ..., {s_r}^0)$
- **WHILE** Relative Residual (RR)$> 0.01 \quad \&\& \quad n <$MaxNumberIter
  - stagnation=100
  - WHILE stagnation> 0.75

Computational Methodology

Algorithm 2

- Initialization $p_{\infty,meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma({s_1}^0, ..., {s_r}^0)$
- **WHILE** Relative Residual (RR)$> 0.01 \quad \&\& \quad n <$MaxNumberIter
  - stagnation=100
  - WHILE stagnation> 0.75
    - Update shape parameters
    - Update material parameters

Computational Methodology

Algorithm 2

- Initialization $p_{\infty,meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma({s_1}^0, ..., {s_r}^0)$
- **WHILE** Relative Residual (RR)$> 0.01 \quad \&\& \quad n <$MaxNumberIter
  - stagnation=100
  - WHILE stagnation> 0.75
    - $\alpha_s \to$ FFP $p_\infty(\lambda^n, \mu^n, \Gamma^n)$
    - Fréchet Derivative, $\frac{\partial p_\infty}{\partial s_i}$ $i = 1, ..., r$
    - Recovery ${s_1}^{n+1}, ..., {s_N}^{n+1}$, n=n+1
    - stagnation$= \frac{p_\infty^n - p_\infty^{n+1}}{p_\infty^n} \cdot 100$
  - Update material parameters

Computational Methodology

Algorithm 2

- Initialization $p_{\infty,meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma({s_1}^0, ..., {s_r}^0)$
- **WHILE** Relative Residual (RR)$>$ 0.01 $\quad \&\& \quad$ $n <$MaxNumberIter
  - stagnation=100
  - WHILE stagnation$>$ 0.75
    - Update shape parameters
  - stagnation=100
  - WHILE stagnation$>$ 0.75
    - $\alpha_m \rightarrow$ FFP $p_\infty(\lambda^n, \mu^n, \Gamma^{n+1})$
    - Fréchet Derivative $\frac{\partial p_\infty}{\partial \lambda}, \frac{\partial p_\infty}{\partial \mu}$
    - Recovery, λ^{n+1}, μ^{n+1} , n=n+1
    - stagnation= $\frac{p_\infty^n - p_\infty^{n+1}}{p_\infty^n} \cdot 100$

Computational Methodology

Algorithm 2

- Initialization $p_{\infty,meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma({s_1}^0, ..., {s_r}^0)$
- **WHILE** Relative Residual (RR)$> 0.01 \quad \&\& \quad n <$MaxNumberIter
  - stagnation=100
  - WHILE stagnation> 0.75
    - Update shape parameters
    - Update material parameters

Computational Methodology

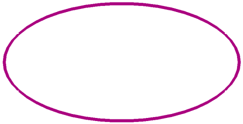
Algorithm 2

- Initialization $p_{\infty,meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma({s_1}^0, ..., {s_r}^0)$
- **WHILE** Relative Residual (RR)$> 0.01 \quad \&\& \quad n <$MaxNumberIter
  - stagnation=100
  - WHILE stagnation> 0.75
    - Update shape parameters
    - Update material parameters
- Increase the frequency (*) $\rightarrow p_{\infty,meas}$

Elliptic coordinate system

Ellipse, Steel media, Algorithm2, Noise Level $= 0\%$

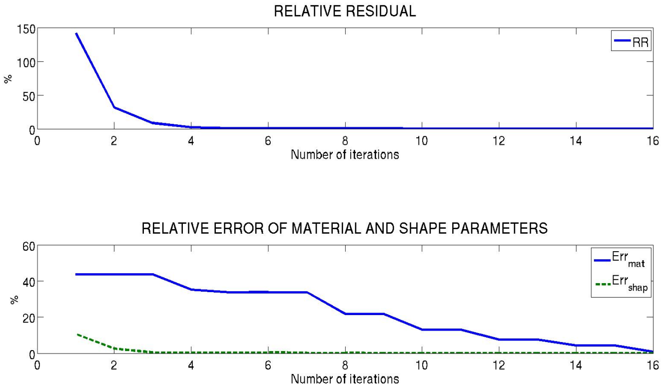
$$s_1, s_2 \rightarrow \Gamma = \{(s_1 \cos\theta, s_2 \sin\theta), \quad \theta \in [0, 2\pi)\}$$



	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

Elliptic coordinate system

Ellipse, Steel media, Algorithm2, Noise Level$= 0\%$

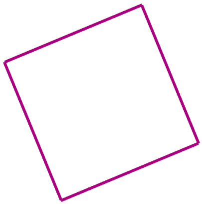


Elliptic coordinate system

	# Iter	k_f	TRR(%)	Err_{mat}(%)	Err_{shap}(%)	α
s_i	1	267	141.63	43.60	10.68	2
s_i	2	267	31.79	43.60	2.58	2
s_i	3	267	8.36	43.60	0.49	2
λ, μ	4	267	2.13	35.10	0.49	$2 \cdot 10^{-30}$
λ, μ	5	267	0.96	33.71	0.49	$2 \cdot 10^{-30}$
λ, μ	6	267	0.84	33.74	0.49	$2 \cdot 10^{-30}$
s_i	7	267	0.84	33.74	0.19	2
λ, μ	8	267	0.49	21.71	0.19	$2 \cdot 10^{-30}$
s_i	9	267	0.34	21.71	0.10	2
λ, μ	10	267	0.27	13.03	0.10	$2 \cdot 10^{-30}$
s_i	11	267	0.18	13.03	0.05	2
λ, μ	12	267	0.14	7.57	0.05	$2 \cdot 10^{-30}$
s_i	13	267	0.09	7.57	0.03	2
λ, μ	14	267	0.07	4.32	0.03	$2 \cdot 10^{-30}$
λ, μ	16	267	0.04	0.76	0.01	$2 \cdot 10^{-30}$

Polygonal parametrization

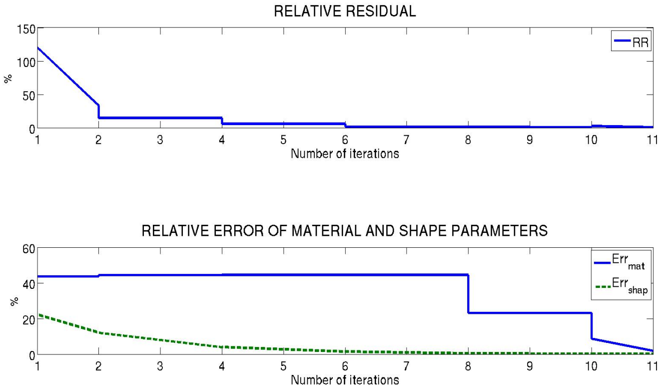
Square, Steel media, Algorithm2, Noise Level$= 0\%$



	Target	In.Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

Polygonal parametrization

Square, Steel media, Algorithm2, Noise Level$= 0\%$



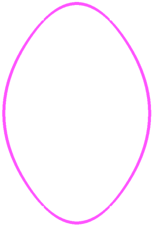
Polygonal parametrization

Square, Steel media, Algorithm2, Noise Level$= 0\%$

	# Iter	k_f	TRR(%)	Err_{mat}(%)	Err_{shap}(%)	α
s_i	1	67	120.28	43.60	22.22	1
s_i	2	83	32.79	43.60	12.13	1
λ, μ	2	100	14.79	44.51	12.13	$3,5 \cdot 10^{-28}$
s_i	4	100	14.78	44.51	3.95	1
λ, μ	4	117	5.86	44.55	3.95	$3,5 \cdot 10^{-28}$
s_i	6	117	5.79	44.55	1.50	1
λ, μ	6	133	1.45	44.54	1.50	$3,5 \cdot 10^{-28}$
s_i	8	133	1.45	44.54	0.48	1
λ, μ	8	150	1.40	23.27	0.48	$3,5 \cdot 10^{-28}$
s_i	10	150	0.89	23.27	0.35	1
λ, μ	10	167	2.39	8.68	0.35	$3,5 \cdot 10^{-28}$
λ, μ	11	167	0.91	1.83	0.35	$3,5 \cdot 10^{-28}$

B-spline representation

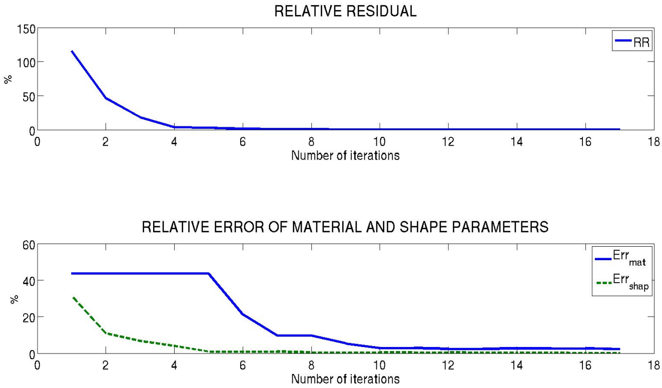
Rugby ball, Steel media, Algorithm2, Noise Level$= 0\%$



	Target	In.Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175

B-spline representation

Rugby ball, Steel media, Algorithm2, Noise Level$= 0\%$



B-spline representation

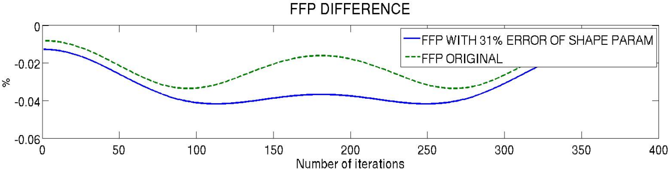
Rugby ball, Steel media, Algorithm2, Noise Level$= 0\%$

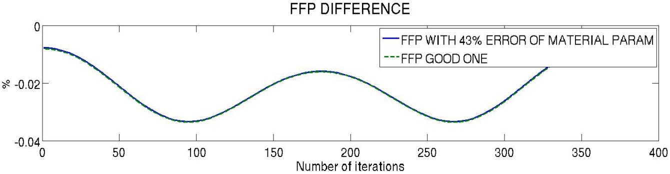
	# Iter	k_f	TRR(%)	Err_{mat}(%)	Err_{shap}(%)	α
s_i	1	66.67	115.47	43.60	31.37	1
s_i	2	100	46.13	43.60	10.90	1
s_i	3	133.33	18.19	43.60	6.71	1
s_i	4	150	3.54	43.60	4.05	1
s_i	5	183.33	2.79	43.60	0.89	1
λ, μ	6	200	1.21	21.19	0.89	$2 \cdot 10^{-28}$
λ, μ	7	200	0.93	9.69	0.89	$2 \cdot 10^{-28}$
s_i	8	200	0.89	9.69	0.39	1
λ, μ	9	233.33	0.21	5.35	0.39	$2 \cdot 10^{-28}$
λ, μ	10	233.33	0.13	2.90	0.39	$2 \cdot 10^{-28}$
s_i	11	233.33	0.12	2.90	0.39	1
λ, μ	12	250	0.11	2.43	0.39	$2 \cdot 10^{-28}$

FFP DIFFERENCE

- $\mathrm{Err}_{mat} = 43\% \rightarrow TRR = 0.95\%$
- $\mathrm{Err}_{shape} = 31\% \rightarrow TRR = 104\%$

FFP DIFFERENCE

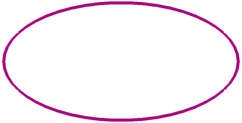




Elliptic coordinate system

Ellipse,Steel media, Algorithm1

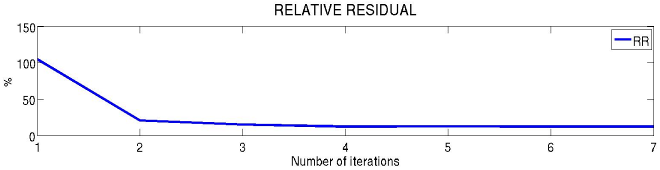
Various regularization parameters, Noise Level$= 2\%$

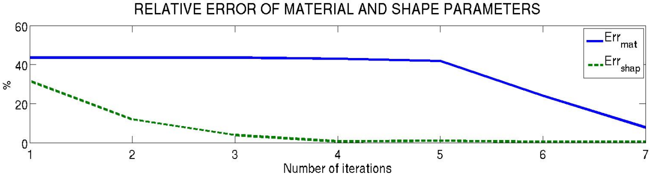


	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

Elliptic coordinate system

Ellipse, Steel media, Algorithm1, Noise Level$= 2\%$





Elliptic coordinate system

Ellipse,Steel media, Algorithm1, Noise Level$= 2\%$

# Iter	k_f	TRR(%)	REMP(%)	RESP(%)	α_{mat}	α_{shape}
1	133	104.76	43.60	31.62	0.5	0.5
2	167	20.76	43.59	11.95	0.05	0.5
3	200	15.17	43.53	3.84	0.005	0.5
4	233	12.25	42.98	0.65	$5 \cdot 10^{-4}$	0.5
5	267	12.46	41.81	0.99	$5 \cdot 10^{-5}$	0.5
6	300	12.32	24.13	0.39	$5 \cdot 10^{-6}$	0.5
7	333	12.32	7.81	0.45	$5 \cdot 10^{-7}$	0.5

Table : Development of Total and Relative Errors

Conclusions of the first stage

- The selection of regularization parameters and frequencies seems to be very sensitive with respect to the problem

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- It is easier to deal with the Algorithm 1
- The first recovery of shape parameters seems to be more efficient.
  STRATEGY: FIRST RETRIEVE THE MORE IFLUENTIAL PARAMETERS.

Recovery of density, shape and location parameters

Influence of density

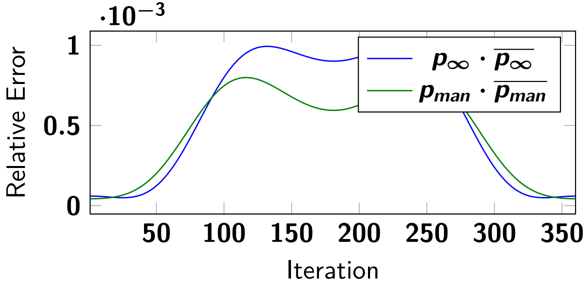

Figure : The difference of FFP's modulus corresponding to **85.18%** of relative error of the density.

Recovery of density, shape and location parameters

Influence of location

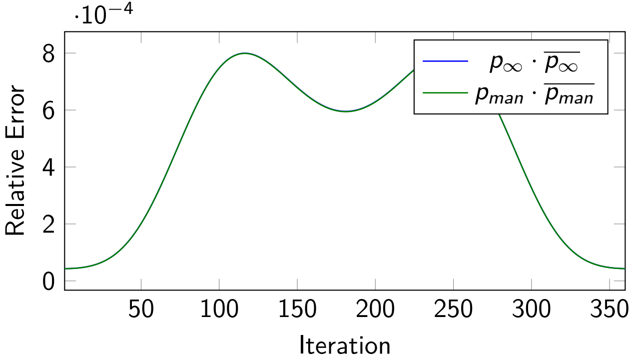

Figure : The difference of FFP's modulus of 0.1% corresponding to 100% of relative error of the location.

Recovery of density, shape and location parameters

Influence of location

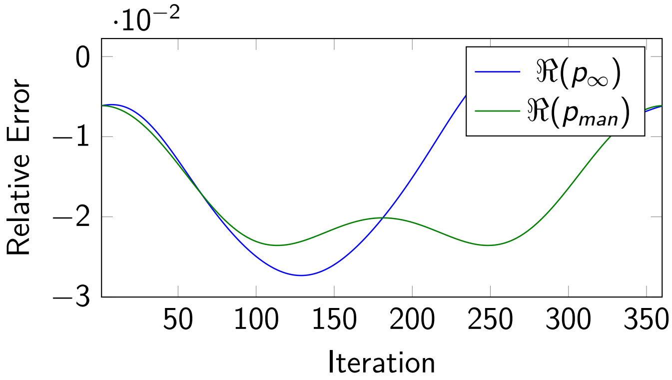

Figure : The difference of FFP's real of 61.77% corresponding to 100% of relative error of the location.

Recovery of density, shape and location parameters

ELLIPSE

Iter	k_f	RR1	RR2	Err_{den}	Err_{shap}	Err_{loc}	α_{den}	α_{shap}	α_{loc}
1	100	180.71	-	92.59	31.62	-	10^{-14}	0.005	-
4	150	19.25	-	33.57	7.51	-	10^{-14}	0.005	-
5	167	3.37	-	18.29	3.41	-	10^{-14}	0.005	-
7	200	0.25	-	0.93	0.06	-	10^{-14}	0.005	-
8	40	-	144.96	-	-	48.09	-	-	0
9	173	-	71.67	-	-	25.13	-	-	0
10	307	-	63.78	-	-	2.01	-	-	0
11	440	-	6.21	-	-	0.06	-	-	0

Table : Relative residual, relative error of shape, density and location parameters for the ellipse case, steel media, with Noise level= 0%.

Recovery of density, shape and location parameters

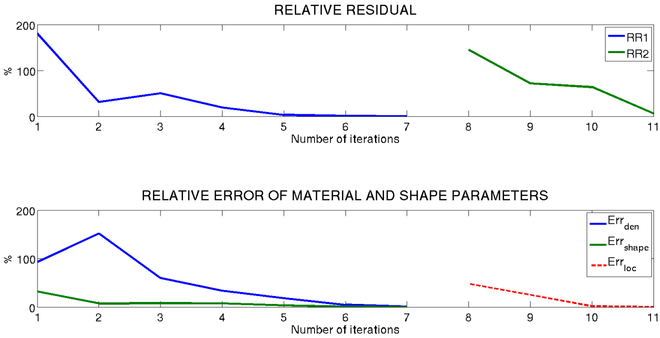

Figure : Relative residual, relative error of shape, density and location parameters for the ellipse case, steel media, with Noise level$=0\%$.

Recovery of density, shape and location parameters

SQUARE, POLYGON

Iter	k_f	RR1	RR2	Err_{den}	Err_{shap}	Err_{loc}	α_{den}	α_{shap}	α_{loc}
1	53	33.30	-	92.59	27.42	-	10^{-10}	20	-
2	120	50.67	-	91.91	44.01	-	10^{-10}	20	-
3	187	56.09	-	94.01	20.41	-	10^{-10}	20	-
4	253	13.64	-	91.22	9.02	-	10^{-12}	20	-
6	387	4.36	-	13.14	4.08	-	10^{-12}	20	-
7	453	1.49	-	2.42	1.09	-	10^{-12}	20	-
9	53	-	108.14	-	-	25.16	-	-	0
10	120	-	24.78	-	-	19.82	-	-	0
11	187	-	31.62	-	-	9.45	-	-	0
13	320	-	14.96	-	-	4.24	-	-	0
16	520	-	7.34	-	-	3.04	-	-	0
18	653	-	10.39	-	-	1.78	-	-	0

Table : Relative residual, relative error of shape, density and location parameters for the square case, steel media, with Noise level= 0%.

Recovery of density, shape and location parameters

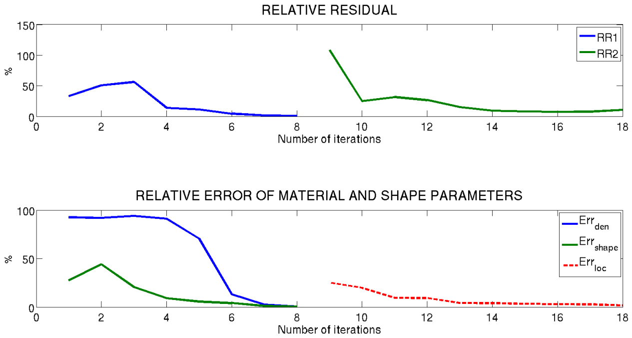

Figure : Relative residual, relative error of shape, density and location parameters for the square case, steel media, with Noise level= 0%.

Recovery of density, shape and location parameters

RUGBY, SPLINES

Iter	k_f	RR1	RR2	Err_{den}	Err_{shap}	Err_{loc}	α_{den}	α_{shap}	α_{loc}
1	67	189.53	-	92.59	43.85	-	10^{-11}	0.1	-
3	133	31.27	-	60.67	42.22	-	10^{-13}	0.1	-
5	200	29.98	-	19.87	26.67	-	10^{-15}	0.1	-
6	233	21.59	-	43.80	15.64	-	10^{-16}	0.1	-
8	300	1.13	-	4.81	0.18	-	10^{-18}	0.1	-
9	333	0.10	-	0.14	0.09	-	10^{-19}	0.1	-
11	200	-	92.15	-	-	38.45	-	-	100
13	467	-	114.77	-	-	20.88	-	-	100
14	600	-	111.31	-	-	16.80	-	-	100
15	733	-	95.22	-	-	3.14	-	-	100

Table : Relative residual, relative error of shape, density and location parameters for the rugby case, steel media, with Noise level$=0\%$.

Recovery of density, shape and location parameters

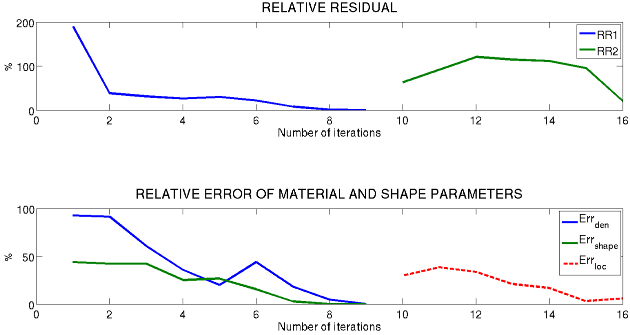

Figure : Relative residual, relative error of shape, density and location parameters for the rugby case, steel media, with Noise level= 0%.

Anisotropic case: recovery of material and shape parameters

Selected material parameters:

- $\rho = 1 \quad [Kg \cdot m^{-3}]$
- $V_p = 0.003 \quad [m \cdot (\mu s)^{-1}]$
- $V_s = 0.0022 \quad [m \cdot (\mu s)^{-1}]$
- $\epsilon = 0.2$
- $\delta = 0.1$

Anisotropic case

Influence of material parameters

Different Relative Errors:

- Rel. Er. of the Real part of the FFP (RE1) $\frac{\|\Re(p_\infty)-\Re(p_{man})\|_2}{\|\Re(p_\infty)\|_2}$
- Rel. Er. of the Imaginary part of the FFP (RE2) $\frac{\|\Im(p_\infty)-\Im(p_{man})\|_2}{\|\Im(p_\infty)\|_2}$
- Rel. Er. of the FFP (RE3) $\frac{\|p_\infty - p_{man}\|_2}{\|p_\infty\|_2}$
- Rel. Er. of the modulus of the FFP (RE4) $\frac{\|p_\infty \overline{p_\infty} - p_m \overline{p_m}\|_2}{\|p_\infty \overline{p_\infty}\|_2}$

Anisotropic case

Influence of density

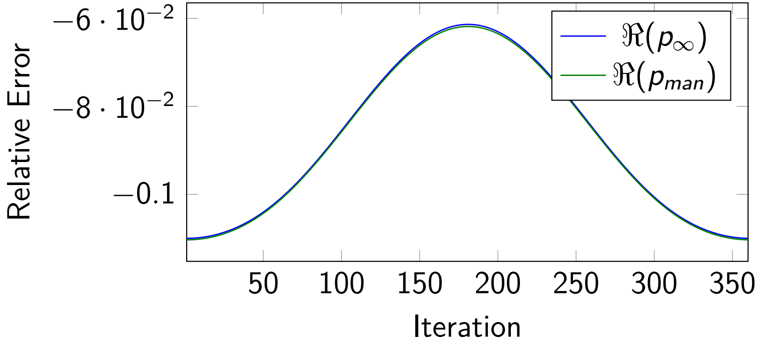

Figure : The difference of FFP's real part corresponding to %100 of relative error of the density.

Anisotropic case

Influence of density

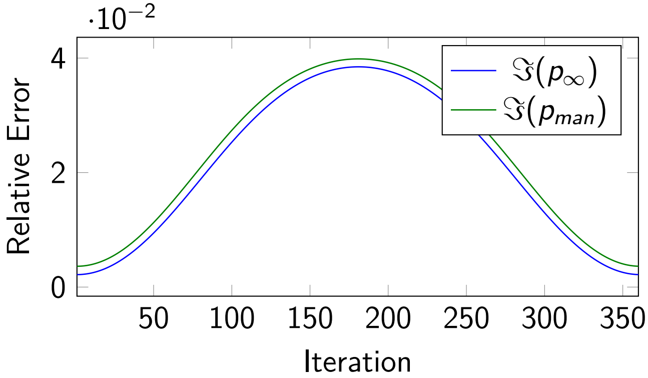

Figure : The difference of FFP's imaginary part corresponding to %100 of relative error of the density.

Anisotropic case

Influence of density

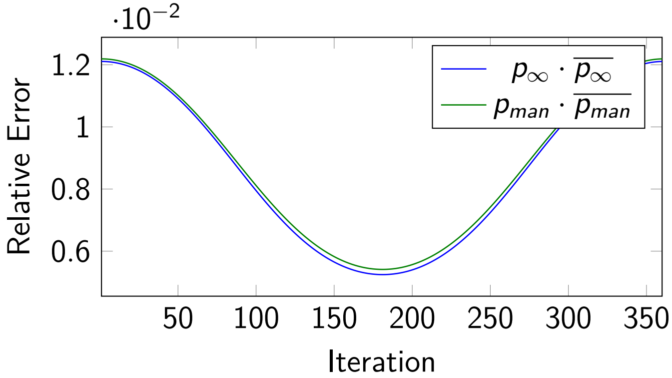

Figure : The difference of FFP's modulus corresponding to %100 of relative error of the density.

Anisotropic case

Influence of density

	RE1(%)	RE2(%)	RE3(%)	RE4(%)
ρ	0.4	7.07	1.94	1.59

Table : The different Relative Errors of the FFP corresponding to the velocity ρ.

Anisotropic case

Influence of V_p

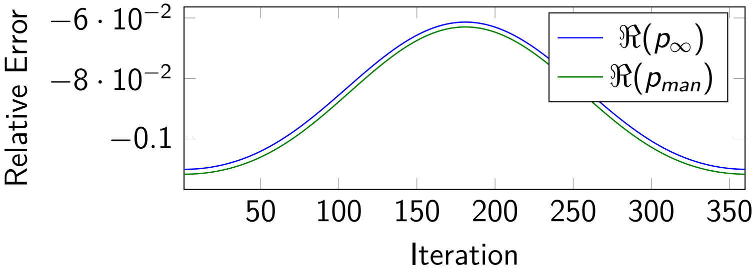

Figure : The difference of FFP's real part corresponding to %100 of relative error of the velocity V_p.

Anisotropic case

Influence of V_p

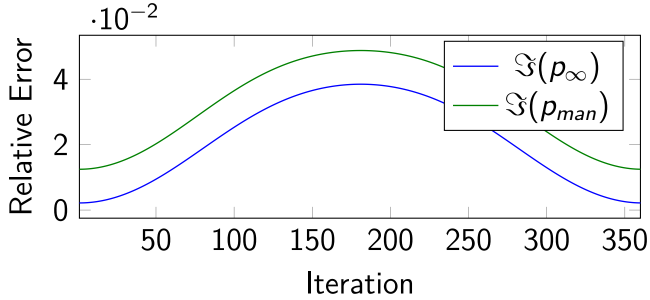

Figure : The difference of FFP's imaginary part corresponding to %100 of relative error of the velocity V_p.

Anisotropic case

Influence of V_p

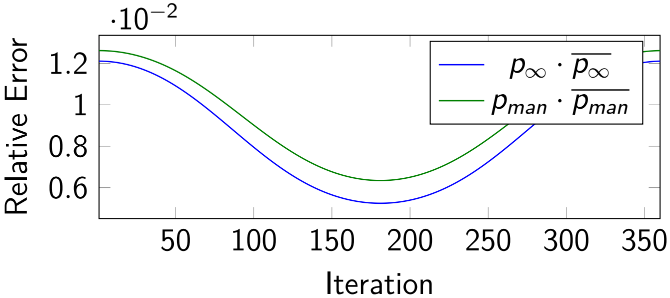

Figure : The difference of FFP's modulus corresponding to %100 of relative error of the velocity V_p.

Anisotropic case

Influence of V_p

	RE1(%)	RE2(%)	RE3(%)	RE4(%)
V_p	2.15	43.01	11.77	10.52

Table : The different Relative Errors of the FFP corresponding to the velocity V_p.

Anisotropic case

Influence of V_s, ϵ and δ

	RE1(%)	RE2(%)	RE3(%)	RE4(%)
V_s	0.54	6.28	1.77	1.69
ϵ	0.04	0.93	0.25	0.17
δ	0.04	0.45	0.13	0.12

Table : The different Relative Errors of the FFP corresponding to the material parameters V_s, ϵ and δ.

Anisotropic case
ELLIPSE

ITER	k_f	$RE1$	Err_ρ	Err_{V_p}	Err_{V_s}	α_ρ	α_{V_p}	α_{V_s}
1	53	13.72	150.86	50.77	122.52	$1\cdot10^{-7}$	$1\cdot10^{-2}$	0.1
2	86	0.89	175.81	44.71	102.81	$1\cdot10^{-7}$	$1\cdot10^{-2}$	0.1
3	120	16.59	47.84	5.88	20.96	$1\cdot10^{-7}$	$1\cdot10^{-2}$	0.1
4	153	0.17	37.87	3.78	17.46	$1\cdot10^{-7}$	$1\cdot10^{-2}$	0.1
5	186	0.33	18.67	3.64	7.94	$1\cdot10^{-7}$	$1\cdot10^{-2}$	0.1
6	220	0.29	9.39	0.45	2.97	$1\cdot10^{-7}$	$1\cdot10^{-2}$	0.1
7	253	0.26	7.81	1.82	2.1	$1\cdot10^{-7}$	$1\cdot10^{-2}$	0.1
8	286	0.3	6.24	1.34	1.65	$1\cdot10^{-7}$	$1\cdot10^{-2}$	0.1
9	320	0.54	4.32	0.93	1.06	$1\cdot10^{-7}$	$1\cdot10^{-2}$	0.1
10	353	$5.97\cdot10^{-2}$	2.82	0.38	0.38	$1\cdot10^{-7}$	$1\cdot10^{-2}$	0.1

Figure : Development of relative residual, relative error of ρ, V_p and V_s in the ellipse case, with constants values for $\epsilon = 0.2, \delta = 0.1$, Noise level=0%.

Anisotropic case

SPLINES

ITER	k_f	$RE1$	Err_{V_p}	Err_{V_s}	reg1	α_{V_p}
1	66.67	1.92	52.1	84.08	$1 \cdot 10^{-2}$	0.1
2	83.33	$9.96 \cdot 10^{-2}$	36.37	64.54	$1 \cdot 10^{-3}$	$1 \cdot 10^{-2}$
3	100	$6.04 \cdot 10^{-2}$	0.43	4.54	$1 \cdot 10^{-4}$	$1 \cdot 10^{-3}$
4	116.67	$3.62 \cdot 10^{-2}$	0.23	0.36	$1 \cdot 10^{-5}$	$1 \cdot 10^{-4}$
5	133.33	$1.71 \cdot 10^{-3}$	$2.33 \cdot 10^{-4}$	$2.63 \cdot 10^{-4}$	$1 \cdot 10^{-6}$	$1 \cdot 10^{-5}$

Figure : Development of relative error 1, relative error of ρ, V_p and V_s in the splines case, with constant values for $\epsilon = 0.2, \delta = 0.1$, Noise level=0%.

Recovery of material and shape parameters

Initial Guess

- $\rho^{(0)} = 1.5 \quad [Kg \cdot m^{-3}]$
- $V_p^{(0)} = 0.006 \quad [m \cdot (\mu s)^{-1}]$
- $V_s^{(0)} = 0.0029 \quad [m \cdot (\mu s)^{-1}]$
- $\epsilon^{(0)} = \epsilon$
- $\delta^{(0)} = \delta$

Anisotropic case

ELLIPSE

ITER	k_f	$RE4$	Err_{V_p}	Err_{shape}	α_{V_p}	α_{shape}
1	53.33	19.04	100	31.62	0.1	1
2	53.33	8.34	52.33	32.53	0.1	0.1
3	53.33	4.74	48.67	18.34	0.1	$1 \cdot 10^{-2}$
4	53.33	23.23	39.33	40.83	0.1	$1 \cdot 10^{-3}$
5	53.33	5.68	17	28.13	0.1	$1 \cdot 10^{-4}$
6	53.33	16.48	41.33	18.09	0.1	$1 \cdot 10^{-5}$
8	53.33	7.62	2.33	20.05	$1 \cdot 10^{-2}$	$1 \cdot 10^{-7}$
9	53.33	3.03	26.67	18.43	$1 \cdot 10^{-3}$	$1 \cdot 10^{-8}$
10	53.33	3.45	15	7.57	$1 \cdot 10^{-3}$	$1 \cdot 10^{-9}$
11	53.33	1.45	0	2.45	$1 \cdot 10^{-3}$	$1 \cdot 10^{-10}$
12	53.33	1.95	10	4.06	$1 \cdot 10^{-3}$	$1 \cdot 10^{-11}$
15	53.33	$4.64 \cdot 10^{-2}$	10	0.38	$1 \cdot 10^{-3}$	$1 \cdot 10^{-14}$

Figure : Development of relative error 4, relative error of V_p and shape parameters in the elllipse case, with constant values for $\epsilon = 0.2, \delta = 0.1$, Noise level=0%.

Anisotropic case
ELLIPSE -Stage 2(a)-

ITER	k_f	$RE2$	Err_ρ	Err_{V_p}	Err_{V_s}	α_ρ	α_{V_p}	α_{V_s}
1	53	$7.95 \cdot 10^{-2}$	50	11.81	4.94	0.1	$1 \cdot 10^{-2}$	0.1
2	53	0.12	50	11.66	7.08	$1 \cdot 10^{-2}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-2}$
3	53	$4.1 \cdot 10^{-2}$	50	11.3	6.34	$1 \cdot 10^{-3}$	$1 \cdot 10^{-4}$	$1 \cdot 10^{-3}$
4	53	$4.21 \cdot 10^{-2}$	50	11.15	6.03	$1 \cdot 10^{-4}$	$1 \cdot 10^{-5}$	$1 \cdot 10^{-4}$
5	53	$4.18 \cdot 10^{-2}$	50	11.13	5.99	$1 \cdot 10^{-5}$	$1 \cdot 10^{-6}$	$1 \cdot 10^{-5}$
6	53	$4.18 \cdot 10^{-2}$	50	11.13	5.99	$1 \cdot 10^{-6}$	$1 \cdot 10^{-7}$	$1 \cdot 10^{-6}$
7	53	$4.18 \cdot 10^{-2}$	50	11.13	5.99	$1 \cdot 10^{-7}$	$1 \cdot 10^{-8}$	$1 \cdot 10^{-7}$
8	53	$4.18 \cdot 10^{-2}$	50	11.13	5.99	$1 \cdot 10^{-8}$	$1 \cdot 10^{-9}$	$1 \cdot 10^{-8}$
9	53	$4.18 \cdot 10^{-2}$	50	11.13	5.99	$1 \cdot 10^{-9}$	$1 \cdot 10^{-10}$	$1 \cdot 10^{-9}$
10	53	$4.18 \cdot 10^{-2}$	50	11.13	5.99	$1 \cdot 10^{-10}$	$1 \cdot 10^{-11}$	$1 \cdot 10^{-10}$

Figure : Development of relative error 2, relative error of ρ, V_p and V_s in the elllipse case, with constant values for $\epsilon = 0.2, \delta = 0.1$, Noise level=0%.

Anisotropic case

ELLIPSE -Stage 2(b)-

ITER	k_f	$RE2$	Err_ρ	Err_{V_p}	Err_{V_s}	α_ρ	α_{V_p}	α_{V_s}
1	53	$7.95 \cdot 10^{-2}$	49.87	9.37	31.81	$1 \cdot 10^{-3}$	1,000	1,000
3	53	$7.78 \cdot 10^{-2}$	40.16	8.56	28.49	$1 \cdot 10^{-5}$	10	10
4	53	0.18	39.48	8.66	28.35	$1 \cdot 10^{-6}$	1	1
7	53	$9.32 \cdot 10^{-2}$	38.42	8.63	28.06	$1 \cdot 10^{-9}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$
8	53	$7.57 \cdot 10^{-2}$	36.42	8.56	27.46	$1 \cdot 10^{-10}$	$1 \cdot 10^{-4}$	$1 \cdot 10^{-4}$
11	53	0.14	1.2	7.86	14.17	$1 \cdot 10^{-14}$	$1 \cdot 10^{-7}$	$1 \cdot 10^{-7}$
15	53	$5.49 \cdot 10^{-2}$	3.64	6.86	13.2	$1 \cdot 10^{-18}$	$1 \cdot 10^{-11}$	$1 \cdot 10^{-11}$

Figure : Development of relative error 2, relative error of ρ, V_p and V_s in the elllipse case, with constant values for $\epsilon = 0.2, \delta = 0.1$, Noise level=0%.

Anisotropic case
SPLINES -Stage 1-

ITER	k_f	$RE4$	Err_{V_p}	Err_{shape}	α_{V_p}	α_{shape}
1	66.67	30.62	100	43.85	1	10
2	83.33	6.22	103	19.95	0.1	1
3	100	1.76	103.33	14.14	$1 \cdot 10^{-2}$	0.1
4	116.67	1.13	101	15.88	$1 \cdot 10^{-2}$	$1 \cdot 10^{-2}$
5	133.33	1.23	93	13.35	$1 \cdot 10^{-2}$	$1 \cdot 10^{-3}$
6	150	1.26	81	11.43	$1 \cdot 10^{-3}$	$1 \cdot 10^{-4}$
7	166.67	1.68	24	9.42	$1 \cdot 10^{-3}$	$1 \cdot 10^{-5}$
8	183.33	0.43	24	0.22	$1 \cdot 10^{-3}$	$1 \cdot 10^{-6}$
9	200	0.16	24	0.58	$1 \cdot 10^{-3}$	$1 \cdot 10^{-7}$
10	216.67	0.19	21	1.01	$1 \cdot 10^{-3}$	$1 \cdot 10^{-8}$
11	233.33	0.26	9	1.57	$1 \cdot 10^{-3}$	$1 \cdot 10^{-9}$
12	250	0.15	6.67	0.59	$1 \cdot 10^{-3}$	$1 \cdot 10^{-10}$

Figure : Development of relative error 4, relative error of V_p and shape parameters in the splines case, with constant values for $\epsilon = 0.2, \delta = 0.1$, Noise level=0%.

Anisotropic case

SPLINES -Stage 2(a)-

ITER	k_f	$RE2$	Err_ρ	Err_{V_p}	Err_{V_s}	α_ρ	α_{V_p}	α_{V_s}
1	66.67	0.51	49.74	0.62	22.08	$1 \cdot 10^{-5}$	100	100
2	83.33	0.48	46.84	0.66	1.2	$1 \cdot 10^{-6}$	100	10
3	100	0.25	45.87	0.67	11.39	$1 \cdot 10^{-7}$	100	1
4	116.67	0.17	37.66	0.67	13.93	$1 \cdot 10^{-8}$	100	0.1
5	133.33	0.16	21.34	0.67	20.34	$1 \cdot 10^{-9}$	100	$1 \cdot 10^{-2}$
6	150	0.17	15.08	0.67	26.66	$1 \cdot 10^{-10}$	100	$1 \cdot 10^{-3}$
7	166.67	0.17	9.03	0.67	31.72	$1 \cdot 10^{-11}$	100	$1 \cdot 10^{-4}$
8	183.33	0.18	7.95	0.67	36.68	$1 \cdot 10^{-12}$	100	$1 \cdot 10^{-5}$

Figure : Development of relative error 2, relative error of ρ, V_p and V_s in the splines case, with constant values for $\epsilon = 0.2, \delta = 0.1$, Noise level=0%.

Anisotropic case

SPLINES -Stage 2(b)-

ITER	k_f	$RE2$	Err_ρ	Err_{V_p}	Err_{V_s}	α_ρ	α_{V_p}	α_{V_s}
1	66.67	0.51	24.54	0.62	19.9	$1 \cdot 10^{-7}$	100	100
2	83.33	0.51	0.21	0.66	16.33	$1 \cdot 10^{-7}$	100	10
3	100	0.33	24.25	0.67	28.13	$1 \cdot 10^{-7}$	100	1
4	116.67	0.19	34.08	0.67	16.01	$1 \cdot 10^{-7}$	100	0.1
5	133.33	0.16	27.08	0.66	18.57	$1 \cdot 10^{-7}$	100	$1 \cdot 10^{-2}$
6	150	0.17	17.77	0.66	24.03	$1 \cdot 10^{-7}$	100	$1 \cdot 10^{-3}$
7	166.67	0.17	11.24	0.66	29.63	$1 \cdot 10^{-7}$	100	$1 \cdot 10^{-4}$
8	183.33	0.18	4.9	0.67	37.63	$1 \cdot 10^{-7}$	100	$1 \cdot 10^{-5}$

Figure : Development of relative error 2, relative error of ρ, V_p and V_s in the splines case, with constant values for $\epsilon = 0.2, \delta = 0.1$, Noise level=0%.

Conclusions

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- The selection of the regularization parameters is very influential in the retrievement of parameters

Perspectives

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- Recovery of location of an anisotropic media

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- Recovery of location of an anisotropic media
- Analytical comparison of ffp's derivatives depending on the parametes
- Shape and material parameters' retrievement of an elastic object immersed in an inhomogeneous media
- Find ideas to increase the influence of the material parameters

MERCI

Cases depending on Lamé coefficients

Steel

$$\lambda = 9.695 \cdot 10^9 \quad N/m^2 \qquad \mu = 7.617 \cdot 10^9 N/m^2$$

$$\mathrm{Err}_{mat}{}^0 = 43.59\%$$
$$\mathrm{Err}\lambda^0 = 48.42\%$$
$$\mathrm{Err}\mu^0 = 34.35\%$$

Aluminium

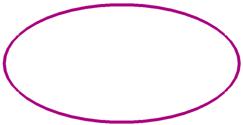
$$\lambda = 5.10878772 \cdot 10^9 \quad N/m^2 \qquad \mu = 2.63165868 \cdot 10^9 \quad N/m^2$$

$$\mathrm{Err}_{mat}{}^0 = 41.26\%$$
$$\mathrm{Err}\lambda^0 = 2.13\%$$
$$\mathrm{Err}\mu^0 = 89.99\%$$

Elliptic coordinate system

Ellipse, Aluminium media, Algorithm1, Noise Level$= 0\%$

$$s_1, s_2 \to \Gamma = \{(s_1 \cos\theta, s_2 \sin\theta), \quad \theta \in [0, 2\pi)\}$$

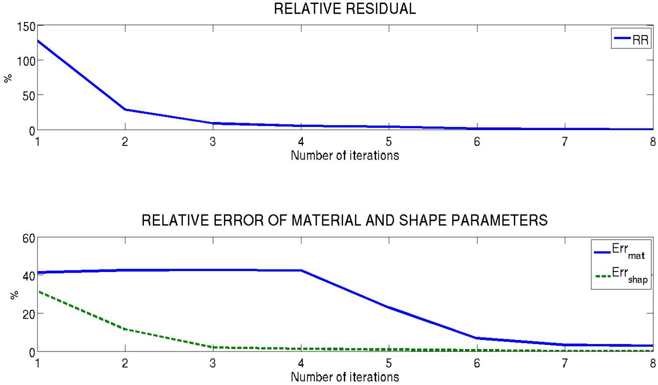


	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

$$k_f = 267, \alpha = 0.5, \alpha^4 = 0.005$$

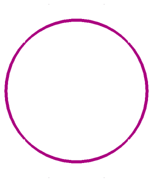
Elliptic coordinate system

Ellipse, Aluminium media, Algorithm1, Noise Level$= 0\%$



Elliptic coordinate system

Circle, Steel media, Algorithm1, Noise Level$= 0\%$

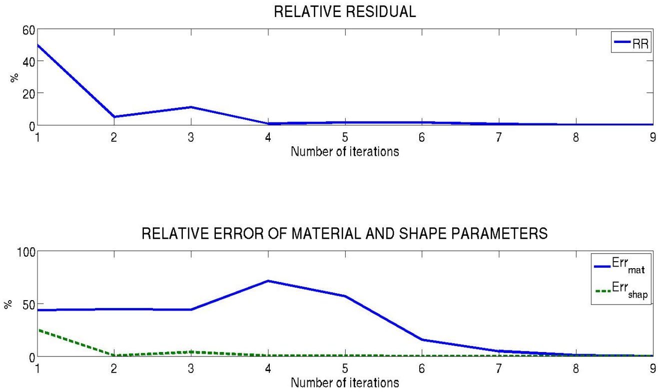


	Target	In. Guess
s_1	0.01	0.0075
s_2	0.01	0.0075

$$k_f = 267, \alpha = 1, \alpha^3 = 5 \cdot 10^{-6}$$

Elliptic coordinate system

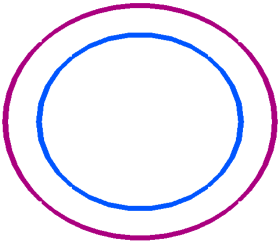
Circle, Steel media, Algorithm1, Noise Level$= 0\%$



Elliptic coordinate system

Circle,Steel media, Algorithm1, Noise Level$= 0\%$

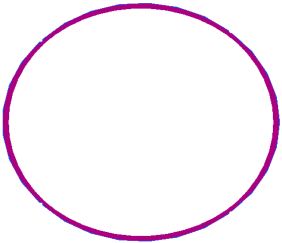
$\#Iter = 0$



Elliptic coordinate system

Circle,Steel media, Algorithm1, Noise Level$= 0\%$

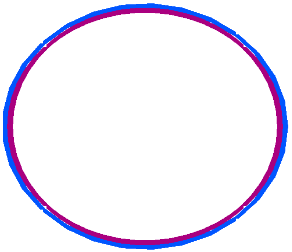
$\#Iter = 1$



Elliptic coordinate system

Circle,Steel media, Algorithm1, Noise Level$= 0\%$

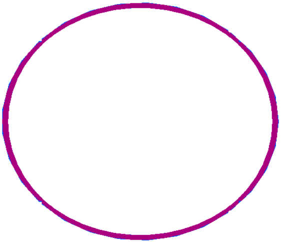
$\#Iter = 2$



Elliptic coordinate system

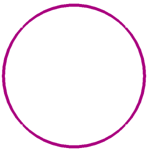
Circle,Steel media, Algorithm1, Noise Level$= 0\%$

$\#Iter = 3$



Elliptic coordinate system

Circle, Aluminium media, Algorithm1, Noise Level$= 0\%$

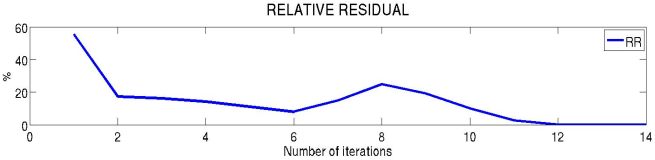


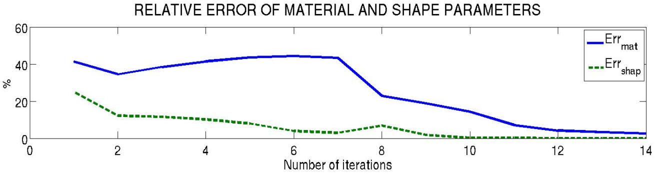
	Target	In. Guess
s_1	0.01	0.0075
s_2	0.01	0.0075

$$k_f = 267, \alpha = 1, \alpha^7 = 0.05$$

Elliptic coordinate system

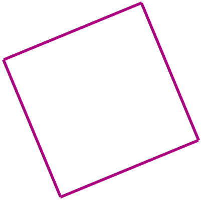
Circle, Aluminium media, Algorithm1, Noise Level$= 0\%$





Polygonal parametrization

Square, Aluminium media, Algorithm1, Noise Level= 0%

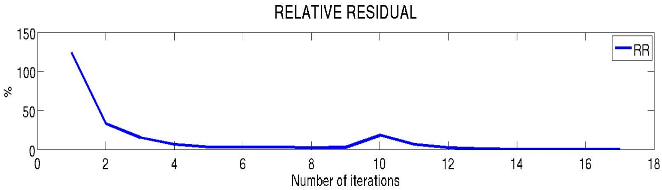


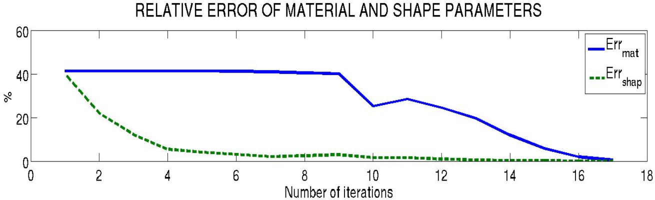
	Target	In.Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

$k_f \in [67, 333], \alpha = 1, \alpha^9 = 0.05$

Polygonal parametrization

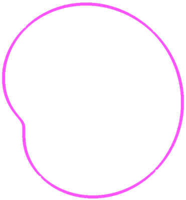
Square, Aluminium media, Algorithm1, Noise Level$= 0\%$





Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level $= 0\%$



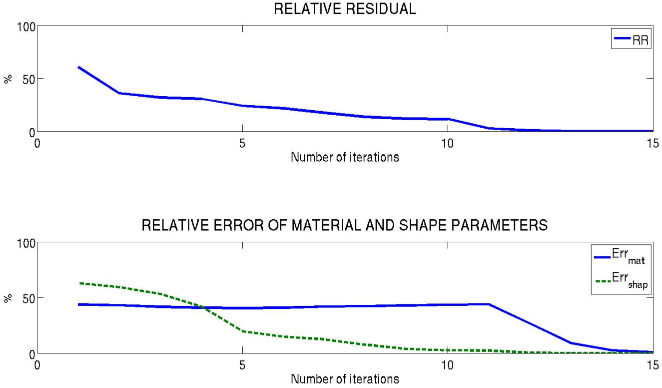
	Target	In.Guess
s_1	0.01	0.0125
s_2	0.007	0
s_3	0.0025	0

Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level= 0%

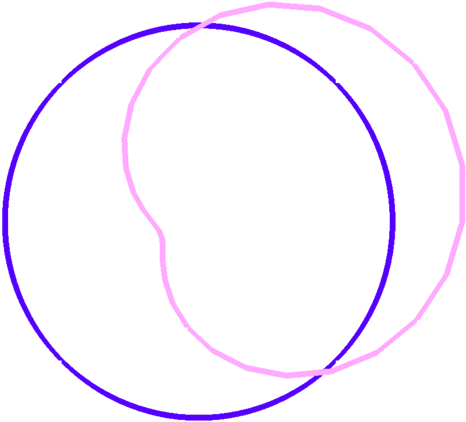
# Iter	k_f	TRR(%)	Err_{mat}(%)	Err_{shap}(%)	α
1	800	60.51	43.60	62.94	12
2	800	35.74	42.94	59.18	12
3	800	31.77	41.68	53.23	12
5	800	23.97	40.30	19.76	12
7	800	17.39	41.82	12.40	12
9	800	11.75	42.86	3.68	12
11	200	2.60	43.78	2.31	10^{-6}
12	200	0.81	26.55	0.28	10^{-6}
13	200	0.08	8.99	0.02	10^{-6}
14	200	0.01	2.46	0.02	10^{-6}
15	200	0.01	0.64	0.01	10^{-6}

Fourier Parametrization



Fourier Parametrization

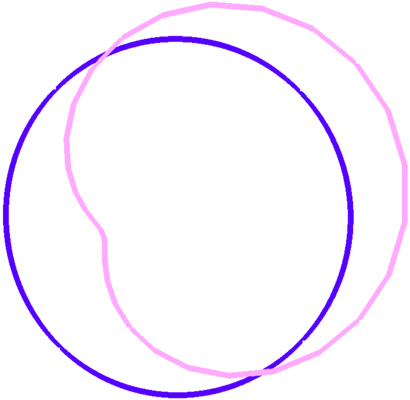
Potato, Steel media, Algorithm1, Noise Level= 0%
#Iter=0



Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level $= 0\%$

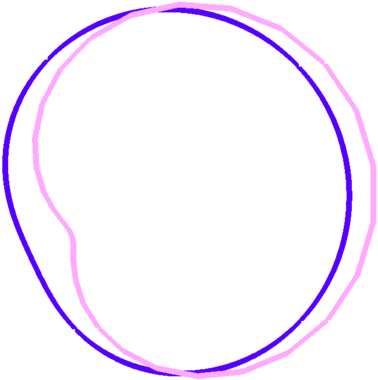
$\#$**Iter=3**



Fourier Parametrization

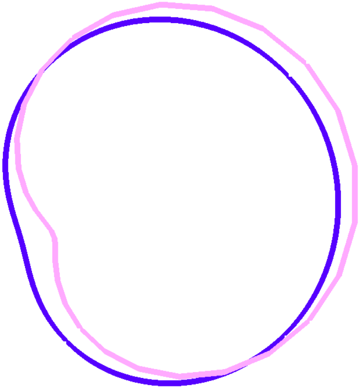
Potato, Steel media, Algorithm1, Noise Level$= 0\%$

#Iter=4



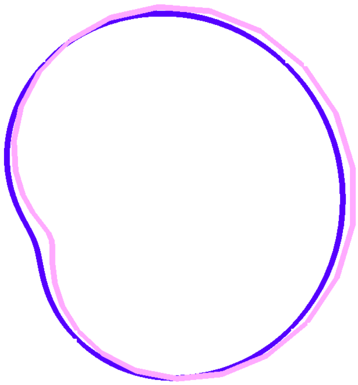
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level$= 0\%$
#Iter=5



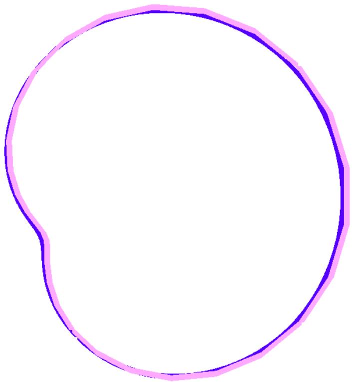
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level$= 0\%$
#Iter=7



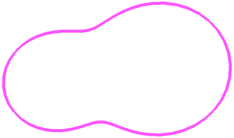
Fourier Parametrization

Potato, Steel media, Algorithm1, Noise Level $= 0\%$
#Iter=9



Fourier Parametrization

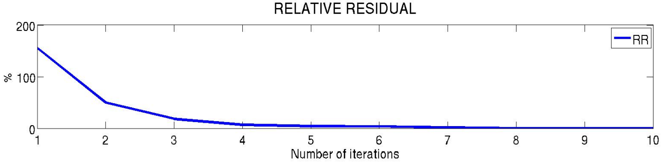
Peanut, Steel media, Algorithm1, Noise Level$= 0\%$

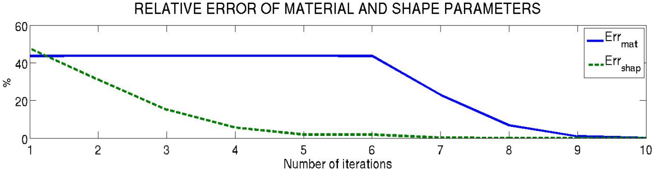


	Target	In.Guess
s_1	0.01	0.0125
s_2	0.002	0
s_3	0.0005	0
s_4	0.004	0
s_5	0.001	0

Fourier Parametrization

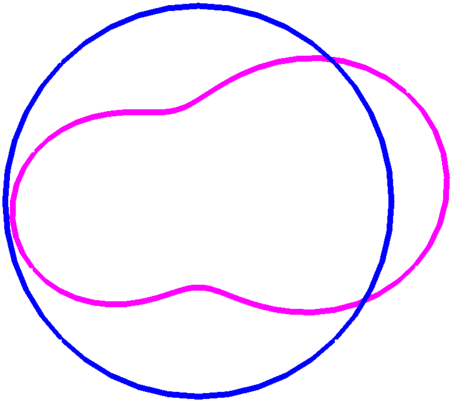
Peanut, Steel media, Algorithm1, Noise Level$= 0\%$





Fourier Parametrization

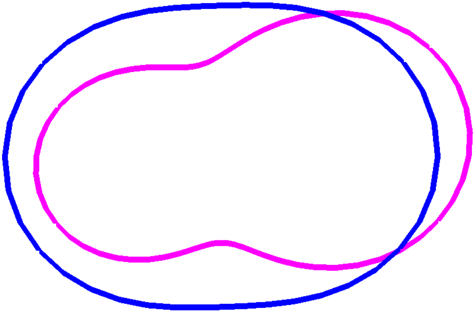
Peanut, Steel media, Algorithm1, Noise Level$= 0\%$ #Iter=0



Fourier Parametrization

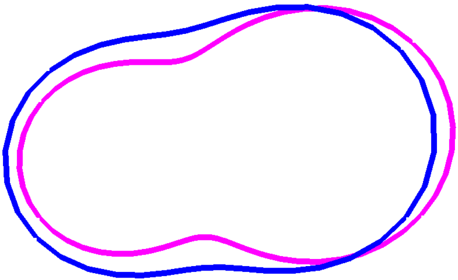
Peanut, Steel media, Algorithm1, Noise Level$= 0\%$

$\#$**Iter=1**



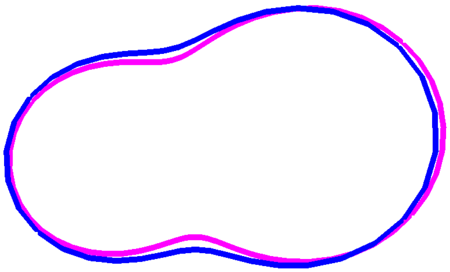
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level= 0% #Iter=2



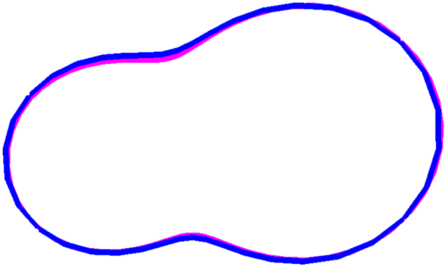
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level$= 0\%$
$\#$**Iter**$=$**3**



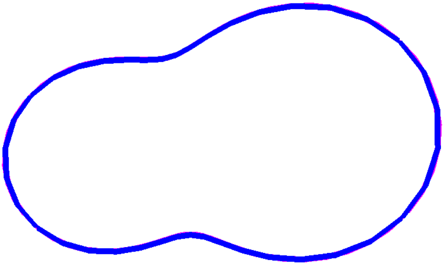
Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level= 0%
#Iter=4



Fourier Parametrization

Peanut, Steel media, Algorithm1, Noise Level$= 0\%$
$\#$**Iter=6**



Computational Methodology

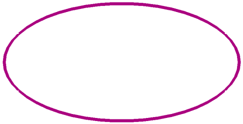
Algorithm 2

- Initialization $p_{\infty,meas}, \omega^0, \alpha^0$ and $\lambda^0, \mu^0, \Gamma^0 \quad : \quad \Gamma^0 = \Gamma({s_1}^0, ..., {s_r}^0)$
- **WHILE** Relative Residual (RR)$>0.01 \quad \&\& \quad n <$MaxNumberIter
  - stagnation=100
  - WHILE stagnation>0.75
    - $\alpha_s \rightarrow$ FFP $p_\infty(\lambda^n, \mu^n, \Gamma^n)$
    - Fréchet Derivative, $\frac{\partial p_\infty}{\partial s_i} \quad i = 1, ..., r$
    - Recovery ${s_1}^{n+1}, ..., {s_N}^{n+1}$, n=n+1
    - stagnation= $\frac{p_\infty^n - p_\infty^{n+1}}{p_\infty^n} \cdot 100$
  - stagnation=100
  - WHILE stagnation>0.75
    - $\alpha_m \rightarrow$ FFP $p_\infty(\lambda^n, \mu^n, \Gamma^{n+1})$
    - Fréchet Derivative $\frac{\partial p_\infty}{\partial \lambda}, \frac{\partial p_\infty}{\partial \mu}$
    - Recovery, λ^{n+1}, μ^{n+1} , n=n+1
    - stagnation= $\frac{p_\infty^n - p_\infty^{n+1}}{p_\infty^n} \cdot 100$
- Increase the frequency (*) $\rightarrow p_{\infty,meas}$

Elliptic coordinate system

Ellipse,Steel media, Algorithm2, Noise Level$= 0\%$

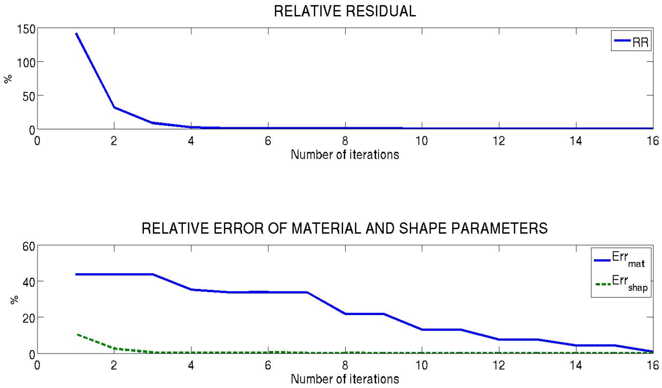
$$s_1, s_2 \to \Gamma = \{(s_1 \cos\theta, s_2 \sin\theta), \quad \theta \in [0, 2\pi)\}$$



	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

Elliptic coordinate system

Ellipse, Steel media, Algorithm2, Noise Level$=0\%$

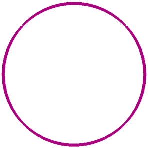


Elliptic coordinate system

	# Iter	k_f	TRR(%)	Err_{mat}(%)	Err_{shap}(%)	α
s_i	1	267	141.63	43.60	10.68	2
s_i	2	267	31.79	43.60	2.58	2
s_i	3	267	8.36	43.60	0.49	2
λ, μ	4	267	2.13	35.10	0.49	$2 \cdot 10^{-30}$
λ, μ	5	267	0.96	33.71	0.49	$2 \cdot 10^{-30}$
λ, μ	6	267	0.84	33.74	0.49	$2 \cdot 10^{-30}$
s_i	7	267	0.84	33.74	0.19	2
λ, μ	8	267	0.49	21.71	0.19	$2 \cdot 10^{-30}$
s_i	9	267	0.34	21.71	0.10	2
λ, μ	10	267	0.27	13.03	0.10	$2 \cdot 10^{-30}$
s_i	11	267	0.18	13.03	0.05	2
λ, μ	12	267	0.14	7.57	0.05	$2 \cdot 10^{-30}$
s_i	13	267	0.09	7.57	0.03	2
λ, μ	14	267	0.07	4.32	0.03	$2 \cdot 10^{-30}$
λ, μ	16	267	0.04	0.76	0.01	$2 \cdot 10^{-30}$

Elliptic coordinate system

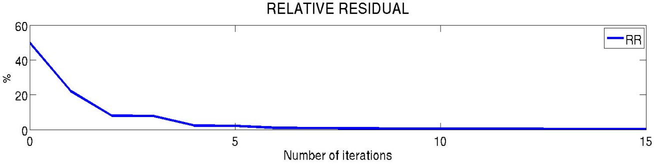
Circle, Aluminium media, Algorithm1, Noise Level $= 0\%$

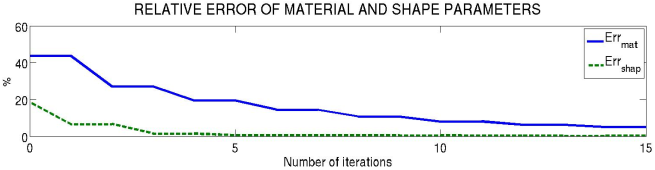


	Target	In. Guess
s_1	0.01	0.0075
s_2	0.01	0.0075

Elliptic coordinate system

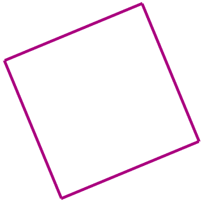
Circle, Steel media, Algorithm2, Noise Level$= 0\%$





Polygonal parametrization

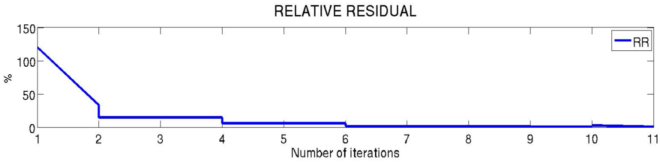
Square, Steel media, Algorithm2, Noise Level$= 0\%$

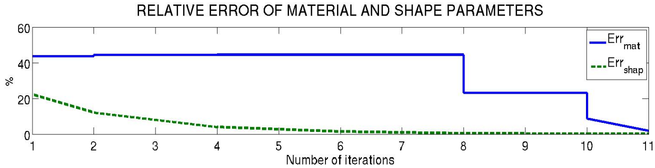


	Target	In.Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

Polygonal parametrization

Square, Steel media, Algorithm2, Noise Level$= 0\%$





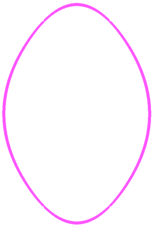
Polygonal parametrization

Square, Steel media, Algorithm2, Noise Level$= 0\%$

	# Iter	k_f	TRR(%)	Err_{mat}(%)	Err_{shap}(%)	α
s_i	1	67	120.28	43.60	22.22	1
s_i	2	83	32.79	43.60	12.13	1
λ, μ	2	100	14.79	44.51	12.13	$3,5 \cdot 10^{-28}$
s_i	4	100	14.78	44.51	3.95	1
λ, μ	4	117	5.86	44.55	3.95	$3,5 \cdot 10^{-28}$
s_i	6	117	5.79	44.55	1.50	1
λ, μ	6	133	1.45	44.54	1.50	$3,5 \cdot 10^{-28}$
s_i	8	133	1.45	44.54	0.48	1
λ, μ	8	150	1.40	23.27	0.48	$3,5 \cdot 10^{-28}$
s_i	10	150	0.89	23.27	0.35	1
λ, μ	10	167	2.39	8.68	0.35	$3,5 \cdot 10^{-28}$
λ, μ	11	167	0.91	1.83	0.35	$3,5 \cdot 10^{-28}$

B-spline representation

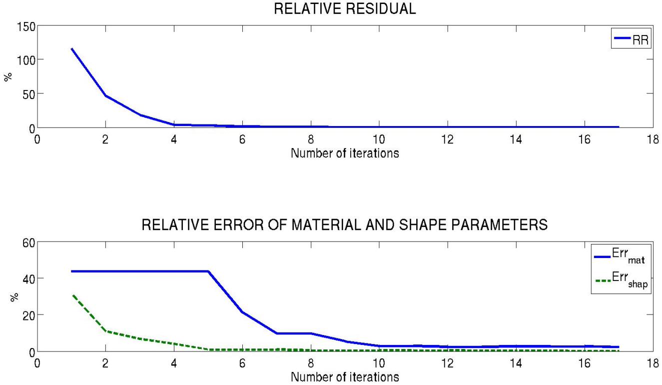
Rugby ball, Steel media, Algorithm2, Noise Level $= 0\%$



	Target	In.Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175

B-spline representation

Rugby ball, Steel media, Algorithm2, Noise Level$= 0\%$



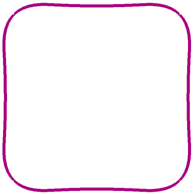
B-spline representation

Rugby ball, Steel media, Algorithm2, Noise Level$= 0\%$

	# Iter	k_f	TRR(%)	$\text{Err}_{mat}(\%)$	$\text{Err}_{shap}(\%)$	α
s_i	1	66.67	115.47	43.60	31.37	1
s_i	2	100	46.13	43.60	10.90	1
s_i	3	133.33	18.19	43.60	6.71	1
s_i	4	150	3.54	43.60	4.05	1
s_i	5	183.33	2.79	43.60	0.89	1
λ, μ	6	200	1.21	21.19	0.89	$2 \cdot 10^{-28}$
λ, μ	7	200	0.93	9.69	0.89	$2 \cdot 10^{-28}$
s_i	8	200	0.89	9.69	0.39	1
λ, μ	9	233.33	0.21	5.35	0.39	$2 \cdot 10^{-28}$
λ, μ	10	233.33	0.13	2.90	0.39	$2 \cdot 10^{-28}$
s_i	11	233.33	0.12	2.90	0.39	1
λ, μ	12	250	0.11	2.43	0.39	$2 \cdot 10^{-28}$

B-spline representation

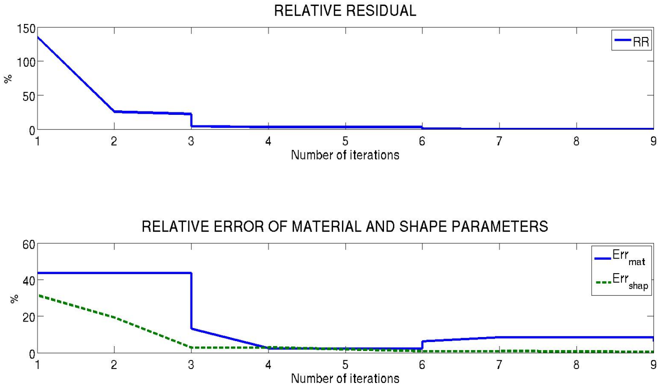
Rounded Square, Steel media, Algorithm2, Noise Level$= 0\%$



	Target	In. Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175
s_5	0.01	0.0175
s_6	0.015	0.0175
s_7	0.01	0.0175
s_8	0.015	0.0175

B-spline representation

Rounded square, Steel media, Algorithm2, Noise Level$= 0\%$



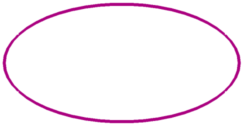
B-spline representation

Rounded square, Steel media, Algorithm2, Noise Level= 0%

	# Iter	k_f	TRR(%)	Err_{mat}(%)	Err_{shap}(%)	α
s_i	1	66.67	135.42	43.60	31.38	1
s_i	2	100	25.52	43.60	19.22	1
s_i	3	133.33	22.42	43.60	2.89	1
λ, μ	4	166.67	4.03	13.09	2.89	$3.5 \cdot 10^{-28}$
λ, μ	5	166.67	3.26	2.46	2.89	$3.5 \cdot 10^{-28}$
s_i	6	166.67	3.19	2.46	0.93	1

Elliptic coordinate system

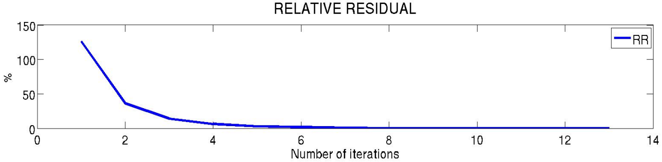
Ellipse,Steel media, Algorithm1,Noise Level$= 2\%$

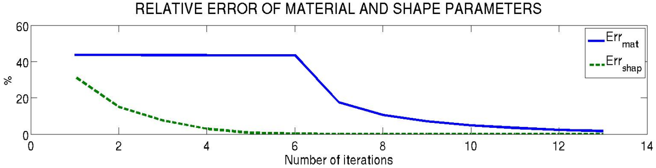


	Target	In. Guess
s_1	0.01	0.0075
s_2	0.005	0.0075

Elliptic coordinate system

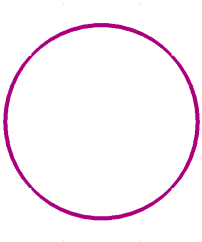
Ellipse, Steel media, Algorithm1, Noise Level$= 2\%$





Elliptic coordinate system

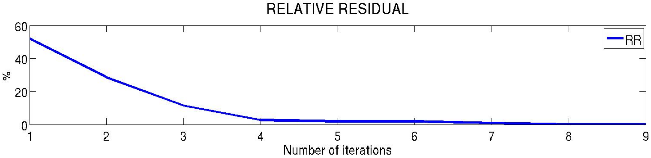
Circle, Steel media, Algorithm1, Noise Level$= 2\%$

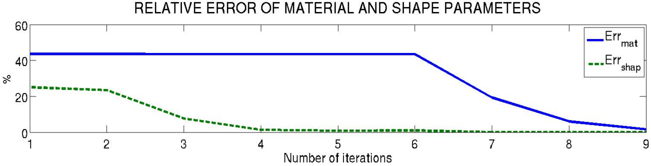


Shape par.	
s_1	0.01
s_2	0.01

Elliptic coordinate system

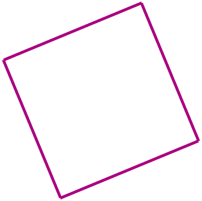
Circle, Steel media, Algorithm1, Noise Level$= 2\%$





Polygonal parametrization

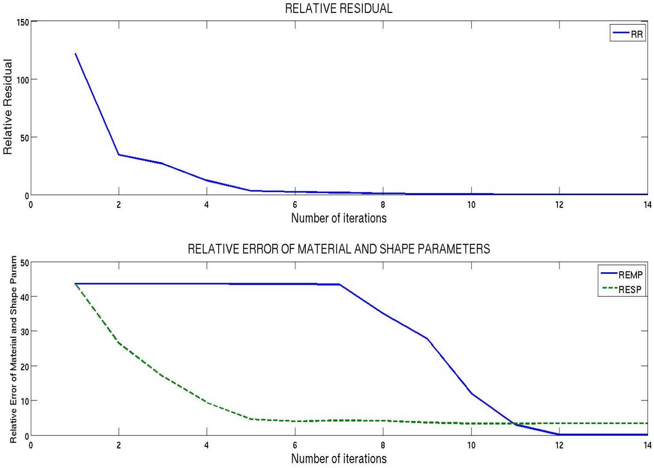
Square, Steel media, Algorithm1, Noise Level$= 2\%$



	Target	In.Guess
s_1	0.010607	0.0175
s_2	0.015	0.0175
s_3	0.010607	0.0175
s_4	0.015	0.0175
s_5	0.010607	0.0175
s_6	0.015	0.0175
s_7	0.010607	0.0175
s_8	0.015	0.0175

Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level$= 2\%$



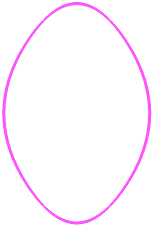
Polygonal parametrization

Square, Steel media, Algorithm1, Noise Level$= 2\%$

# Iter	k_f	TRR(%)	Err$_{mat}$(%)	Err$_{shap}$(%)	α
1	67	121.74	43.60	43.85	2
2	100	34.31	43.61	26.47	2
3	133	26.70	43.61	16.92	2
4	167	12.09	43.60	9.35	2
5	200	3.18	43.55	4.61	2
6	233	2.34	43.50	3.97	2
7	267	1.61	43.42	4.23	$2 \cdot 10^{-6}$
8	300	1.09	34.92	4.09	$2 \cdot 10^{-6}$
9	333	0.47	27.75	3.64	$2 \cdot 10^{-6}$
10	367	0.25	12.03	3.35	$2 \cdot 10^{-6}$
11	400	0.09	3.02	3.33	$2 \cdot 10^{-6}$
12	433	0.02	0.12	3.37	$2 \cdot 10^{-6}$

B-spline representation

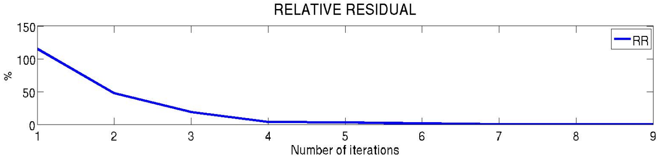
Rugby ball, Steel media, Algorithm1, Noise Level $= 2\%$

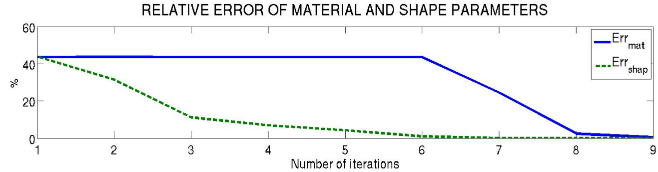


	Target	In.Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175

B-spline representation

Rugby ball, Steel media, Algorithm1, Noise Level= 2%



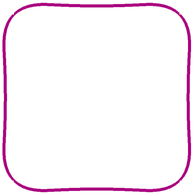


Rugby ball, Steel media, Algorithm1, Noise Level$= 2\%$

# Iter	k_f	TRR(%)	$Err_{mat}(\%)$	$Err_{shap}(\%)$	α
1	67	114.95	43.60	43.85	1
2	100	47.13	43.61	31.42	1
3	133	18.55	43.60	11.07	1
4	167	3.61	43.60	6.85	1
5	200	2.83	43.60	4.14	1
6	233	1.22	43.59	0.90	10^{-7}
7	266	0.32	24.43	0.10	10^{-7}
8	300	0.04	2.42	0.01	10^{-7}
9	333	0.01	0.46	0.00	10^{-7}

B-spline representation

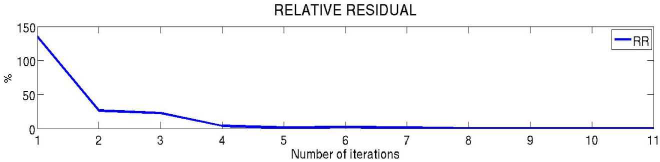
Rounded Square, Steel media, Algorithm1, Noise Level$= 2\%$

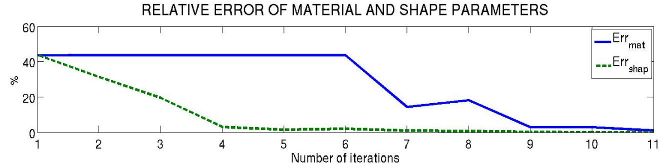


	Target	In. Guess
s_1	0.01	0.0175
s_2	0.015	0.0175
s_3	0.01	0.0175
s_4	0.015	0.0175
s_5	0.01	0.0175
s_6	0.015	0.0175
s_7	0.01	0.0175
s_8	0.015	0.0175

B-spline representation

Rounded Square, Steel media, Algorithm1, Noise Level$= 2\%$





Rounded Square, Steel media, Algorithm1, Noise Level $= 2\%$

# Iter	k_f	TRR(%)	Err_{mat}(%)	Err_{shap}(%)	α
1	67	135.73	43.60	43.85	1
2	100	26.34	43.61	31.34	1
3	133	22.63	43.64	19.51	1
4	167	3.94	43.72	3.00	1
5	200	1.31	43.71	1.51	1
6	233	2.21	43.70	1.96	10^{-7}
7	267	1.23	14.29	0.92	10^{-7}
8	300	0.28	18.13	0.64	10^{-7}
9	333	0.10	2.82	0.18	10^{-7}